COLLECTED WORKS

BY ALEXEY ZYKIN

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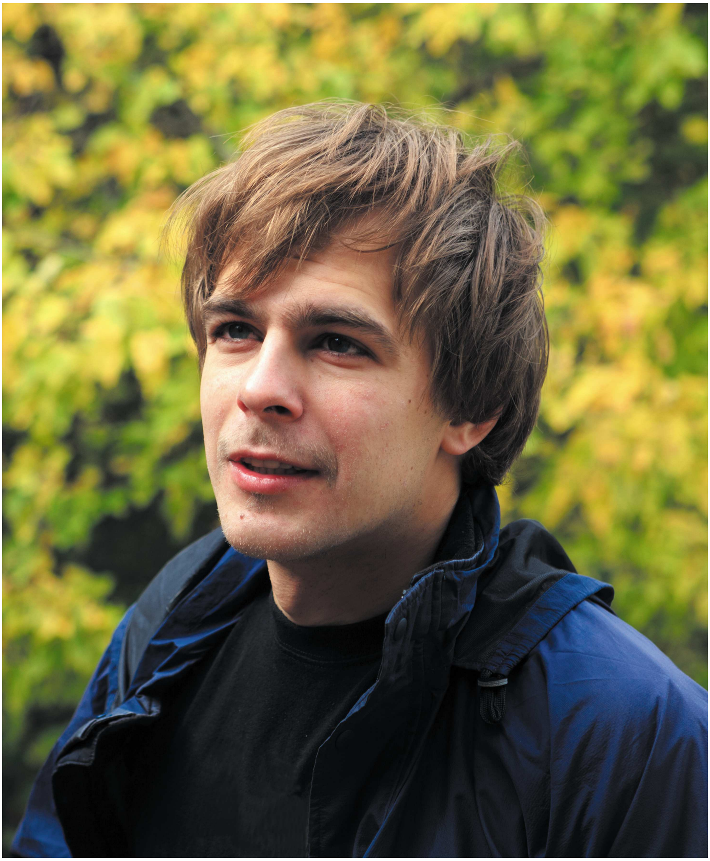

Alexey Zykin

(13.06.84—22.04.17)

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Collected Works

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Alexey Zykin (1984—2017)

On June 13, 2019, Alexey Zykin, a brilliant mathematician and professor, our dear friend and colleague, would have turned 35.

Alexey was born in Moscow in 1984. His parents are not mathematicians: his father Ivan Semenovich is a professor of legal studies, one of the leading Russian specialists in private international law and public commercial international law, his mother Yulia Ivanovna is an economist specializing in foreign trade.

At the age of 14 Alexey was admitted to one of the best high schools in Moscow, School No. 57, to a class majoring in mathematics. The core math courses in his class were taught by Rafail Gordin and Petr Sergeev. Among those who taught them advanced mathematics, were such top-level research mathematicians as Alexander Kuznetsov and Valentina Kirichenko. As early as in his school years, Alexey became interested in number theory; in particular, he thoroughly studied the famous textbook by Ireland and Rosen.

In 2000, during his final high school year, Alexey was admitted to the Independent University of Moscow, and in 2001, after graduation from high school, to the Mechanics and Mathematics department of Moscow State University, together with most of his classmates. During the first year of his undergraduate studies, he started doing his own research under the supervision of Professor Michael Tsfasman. Alexey's first research paper, "Brauer—Siegel and Tsfasman—Vlăduţ theorems for almost normal extensions of global fields" was published when he was on the fourth year of his undergraduate studies. During all of his undergraduate and graduate years, Alexey continued working under the supervision of Michael Tsfasman, who had the most significant influence on both the subject-matter and the style of Alexey's mathematical research.

After graduating *cum laude* from the Independent University in 2005 and from the Moscow State in 2006, Alexey, a graduate student of Steklov Mathematical Institute, obtained a scholarship for graduate studies from the Government of France. This scholarship entitled him to spend six months per year in France. Thus Alexey became a graduate student at the University of Aix—Marseille II in Luminy, near Marseille, under joint supervision of Tsfasman and Serge Vlăduţ. In June 2009 he defended his Ph.D. thesis in France, and then, in October 2010, in Russia.

At the present time young mathematicians rarely find a permanent academic job right after their Ph.D. thesis, usually spending several years as a postdoc before that. But this was not the case with Alexey: in 2009 he obtained an Assistant Professor position at the newly created Department of Mathematics of Higher School of Economics, in Moscow. For several years he was the youngest faculty member of this department, and certainly among the most active ones. He taught numerous courses, compulsory and elective ones, both at HSE and at the Independent University. He also organized numerous seminars and supervised several undergraduate students, who did their research in various areas, not only in number theory. For instance, one of HSE undergraduates, Dmitry Grischenko, published a paper on the mathematics of origami, written under Alexey's supervision. Every year in 2011—13 Alexey obtained the HSE Award for Teaching Excellence, based on the results of student polls.

Alexey was a gifted administrator and organizer of various events. Among his main achievements in this area, let us mention the creation of a summer school "Algebra and Geometry" in Yaroslavl. This school was launched in 2011; since then, it is being held every year. It is aimed at senior undergraduate students and Ph.D. students; in a certain sense, this makes it a successor of the famous summer school "Contemporary Mathematics" in Dubna, aimed at senior high school students and first-second year undergrads. Alexey also participated in the Dubna school, first as a student and later as an instructor. Also in 2012—13 Alexey was the head of the Laboratory of algebraic geometry and its applications at the Higher School of Economics.

In 2014, at the age of less than 30 (again an exceptional case!), Alexey obtained a permanent professor position in France: more precisely, in the most remote part of it, at the University of French Polynesia, in Papeete, Tahiti. But, despite being physically present in the opposite point of the globe, he kept participating in the mathematical life in Moscow: he kept contacts with his Moscovite colleagues, supervised students and came to Russia every summer to participate in the Yaroslavl school. A couple of months before death, he was appointed head of GAATI (Algebraic Geometry and Applications to Information Theory) research group at his university.

Alexey's interests were not at all limited to mathematics: he was a polymath, had a good knowledge of literature and music, fluently spoke English and French, had a taste in wines and good cuisine, loved to travel (it seems that he had been nearly everywhere in the world, from the

Himalayas to Kilimanjaro), practiced sports, especially rock climbing and diving…

On April 22, 2017, Alexey Zykin, his wife Tatiana Makarova and a diving instructor, Gilles Demée, died while exploring an underwater cave in Ahe atoll in the Tuamotus, French Polynesia.

Alexey left a broad scientific heritage that consists of 15 published works (two of them being published posthumously). For his results he obtained prestigious prizes: Moscow Mathematical Society Award (2011), "Dynasty" Foundation Award (2010), and others. His research was mostly in asymptotic theory of global fields and arithmetic varieties. This part of modern mathematics is developed extensively and lies in between analytic number theory, algebraic number theory, and algebraic geometry. Its foundations were established by Michael Tsfasman and Serge Vlăduţ.

Let us say some words about this domain. Important mathematical objects of study are systems of polynomial equations with integer coefficients or, more generally, arithmetic varieties. Note that one-dimensional case is just the theory of global fields. To an arithmetic variety one associates a complex-analytic function in one variable, called its zeta function. There are deep relations between analytic properties of the zeta function and properties of the arithmetic variety. Each new result towards this relationship is a true breakthrough. It turns out that, given an infinite family of global fields or arithmetic varieties, in a wide range of cases the limits of the zeta functions have many remarkable properties, that reflect many features of varieties themselves. Here, an important condition on the family is its being asymptotically exact. However, this is not a very restrictive condition, since any infinite family contains an asymptotically exact subfamily. These are the questions studied in the asymptotic theory of global fields and, more generally, of arithmetic varieties. Another source of interest in these investigations is provided by numerous applications in coding theory and cryptography. Zykin made fundamental contributions in these domains.

All of Zykin's papers are wonderfully written, with perfect style and crystal clarity. Key issues of the reasonings are explained in minute detail. Many papers contain lists of problems for further research. Besides their high scientific value, Zykin's papers can serve as an excellent introduction to the asymptotic theory of global fields and arithmetic varieties for a wide range of mathematicians.

Now let us describe Zykin's papers in more detail.

In the paper [1] he proves a strengthening of the classical Brauer—Siegel theorem. Namely, given a tower of number fields $\{K_i\}$, one considers the limit of the ratio $\log(h_{K_i}R_{K_i})/g_{K_i}$, where $g_{K_i} = \log\sqrt{|D_{K_i}|}$ and h_{K_i}, R_{K_i}, D_{K_i} denote the class number, the regulator, and the discriminant of K_i, respectively. The classical Brauer—Siegel theorem claims that if either all the fields K_i are normal over $\mathbb{Q}$, or the Generalized Riemann Hypothesis (GRH) holds for them, then under certain additional conditions the limit equals 1. In Zykin's paper under the same conditions (the so-called asymptotically bad case in the terminology of Tsfasman—Vlăduţ) an analogous result is proved for the tower of almost normal fields (Theorem 1). Also, he observes that results of Tsfasman—Vlăduţ on the generalization the Brauer—Siegel theorem give an analogous statement for asymptotically good towers as well. Besides, under the assumption of GRH, Zykin constructs new examples of towers with the limit of the Brauer—Siegel ratio being closer to the lower bound than in the examples known before (Theorem 4).

The article [2] is a survey of results by Tsfasman, Vlăduţ, Zykin, and Lebacque on families of global fields and of results by Kunyavskii—Tsfasman and Hindry—Pacheko on families of elliptic curves over function fields and number fields, respectively. Besides, for an asymptotically exact family $\{X_i\}$ of varieties of dimension d over a finite field, a theorem on the limit of residues at $s = d$ of the zeta functions $\zeta_{X_i}(s)$ (Theorem 3.2) is proved.

Further, in [3] Zykin considers an elliptic curve E over a function field K and a tower $\{K_i\}$ of extensions of K, and studies the asymptotic behavior of L-functions $L_{E_i}(s)$ of elliptic curves $E_i = E \times_K K_i$ over the fields K_i. He obtains a statement on the limit of the leading coefficients in decompositions of $L_{E_i}(s)$ in Taylor series at $s = 1$ (Theorem 2, part 3). Note that it makes sense not only to consider the limits of residues and leading coefficients of zeta and L-functions, but also to study the asymptotic behavior of the functions themselves as functions of a complex variable in a suitable domain. Zykin presents in [3] a statement on the asymptotic behaviour of functions $\log L_{E_i}(s)$ on the domain $\operatorname{Re} s > 1$ (Theorem 2, part 2). Complete proofs of these statements are given in [11].

The note [4] contains a brief statement of results of the paper [5]. The main results of [5] are as follows. For an asymptotically exact family of number fields $\{K_i\}$, one studies the asymptotic behavior of the logarithms of zeta functions $\log\zeta_{K_i}(s)$. In all the results GRH is assumed. It is proved that in the domain $\operatorname{Re} s > 1/2$ the limit of functions $\log((s-1)\zeta_{K_i}(s))/g_{K_i}$

is equal to the logarithm of the so-called limit zeta function $\log\zeta_{\{K_i\}}(s)$ of the family $\{K_i\}$ introduced before by Tsafsman and Vlăduţ (Theorem 2). This gives a conceptual explanation of the generalized Brauer—Siegel theorem and also, as an application, leads to new results on limits of the Euler—Kronecker constants, which are analogs for numbers fields of known results by Ihara for function fields (Corollary 1). Besides, Zykin obtains a nontrivial upper bound for the limit of the logarithm of the leading coefficients in decompositions of $\zeta_{K_i}(s)$ at $s=1/2$ (Theorem 3). In the proofs, bounds on the logarithmic derivative of zeta functions on the critical strip and results on the asymptotic behavior of zeroes of the zeta functions on the critical line in families of number fields are used.

The short article [6] announces the results from [7]. Zykin's joint paper [7] with Gilles Lachaud and Christophe Ritzenthaler gives an answer to an important question of the great mathematician J.-P. Serre: how to determine whether a principally polarized abelian threefold (A, a) over a field $k\subset\mathbb{C}$ is the Jacobian of a curve over k? To do that, the authors use a certain arithmetic invariant $\chi_{18}(A, a, \omega)\in k$, where ω is a basis in the space of regular 1-forms on A. This invariant is expressed in terms of an analytic Siegel modular form $\widetilde{\chi}_{18}$ and allows one to distinguish abelian threefolds which are isomorphic over a quadratic extension of the ground field. Combining this with a result of Serre, the authors obtain an answer to the initial question (Theorem 1.3.3). Essentially, the answer is reduced to the statement that (A, a) is the Jacobian of a non-hyperelliptic curve if and only if $\chi_{18}(A, a, \omega)$ is a non-zero square in k. Besides, in the paper, one can find a new simple and nice proof of the classical formula of Klein, which is closely related to the above question and has to do with the equality $\mathrm{Disc}(F)^2=\chi_{18}(A, a, \omega)$, where $F(x_1, x_2, x_3)$ is a smooth homogeneous polynomial of degree 4 and A is the Jacobian of the corresponding smooth plane quartic with the natural polarization a and a natural basis of 1-forms ω (Theorem 2.2.3).

Another short note [8] contains statements of results from [9]. In Zykin's joint paper [9] with Philippe Lebacque, the authors investigate the asymptotic behaviour of the logarithmic derivatives $Z_K(s)=\zeta'_K(s)/\zeta_K(s)$ of the zeta functions $\zeta_K(s)$ of global fields K. For all results on number fields the authors assume GRH. Note that, since the zeta function $\zeta_K(s)$ is given by an infinite product, the function $Z_K(s)$ is determined by an infinite series. In [9], the authors first prove a fine explicit bound on the approximation error in the expression of $Z_K(s)$ as an infinite series on the domain $\operatorname{Re} s>1/2$ (Theorems 1.1 and 1.2). In the proof of this result, they demonstrate their extraordinary ability to use complicated analytic techniques and Weil's explicit formulae. In particular, the bound leads to a new proof of the so-called basic inequality in the asymptotic theory of global fields (Remarks 2.2 and 3.2). Then, the above bound is applied to the logarithmic derivative $Z_{\{K_i\}}(s)$ of the limit zeta function $\zeta_{\{K_i\}}(s)$ of an asymptotically exact family of global fields $\{K_i\}$. Namely, they obtaine a new bound on the approximation error in the expression of $Z_{\{K_i\}}(s)$ as an infinite series on the domain $\mathrm{Re}\, s > 1/2$ (Corollary 1.3). Besides, the authors find an explicit bound for the approximation error in the expression of the value $Z_{\{K_i\}}(1/2)$ as an infinite series (Theorem 1.4). Finally, this implies an explicit bound on the approximation error in the expression of the value $\log \zeta_{\{K_i\}}(1)$ as an infinite series (Corollary 1.5). The latter bound is a far reaching strengthening of the classical Brauer—Siegel theorem.

The joint paper with Lebacque [10] is a survey of the asymptotic theory and serves as a wonderful introduction to it. First, the foundations of this theory established by Tsfasman and Vlăduţ are explained: limit invariants, asymptotically exact families, the basic inequality, which is a far reaching generalization of both the Odlyzko—Serre estimates and the Drinfeld—Vlăduţ inequality (Section 2). Then they discuss generalizations of the Brauer—Siegel theorem obtained by Tsfasman, Vlăduţ, and the authors of the paper, the asymptotic behaviour of zeta functions and their zeroes, and also the amazing relations between these topics and the limit zeta function (Section 3). Then they give examples of towers of function fields that are asymptotically optimal, that is, that reach the bound from the basic inequality (Section 4). Such towers correspond to iterated coverings of curves over a finite field that have the maximal possible number of points. In the survey, there is a wide range of examples of asymptotically optimal towers constructed by Ihara, Tsfasman, Vlăduţ, Zink, Elkies, Garcia, and Stichtenoth. Besides, the authors discuss a higher-dimensional generalization of the asymptotic theory of global fields (Section 5). Namely, they formulate results by Lachaud—Tsfasman that generalize the basic inequality to asymptotically exact families of varieties over a finite field. Also, conjectural generalizations of the Brauer—Siegel theorem to the case of abelian varieties over a function field proposed by Kunyavskii—Tsfasman and Hindry—Pacheko are stated. Finally, they briefly mention the theory of abstract L-functions over a finite fields, which is explained in more detail in the next paper.

The paper [11] contains the foundations of the general asymptotic theory of varieties over finite fields and over function fields. Main results in the asymptotic theory of function fields are generalized to the case of infinite families of abstract zeta and L-functions over a finite field. To do this, Zykin carefully analyzes which general arithmetic properties of zeta functions lead to these results. It turns out that, actually, it is enough to require only an analog of the statement on absolute values of Frobenius eigenvalues and also an analog of the fact that numbers of points are nonnegative, or, even more, its weakening given by the so-called asymptotic very exactness of families (Definition 3.10). Having developed explicit formulae in this abstract set-up (Section 2.2), the author deduces from them many nontrivial results. Let us note a theorem on the limit distribution of zeroes (Theorem 4.1), a version of the generalized Brauer—Siegel theorem on the asymptotic behavior of zeta functions (Theorems 5.5 and 5.9), and a version of the basic inequality (Theorems 6.1 and 6.6). As an application, Zykin obtains a result on the asymptotic behaviour of higher Euler—Kronecker constants for families of function fields, which strengthens known results by Ihara (Corollary 5.16), and also obtains a new proof of the basic inequality (Remark 6.5). Moreover, all results are well illustrated by applications to families of elliptic curves over function fields (Corollary 4.9, Theorem 5.27).

In the short and elegant paper [12], the author studies families of primitive cusp forms f_i of level N_i and weight k_i such that the number $N_i k_i^2$ tends to infinity. For each form f_i, he considers its L-function $L_{f_i}(s)$ with the argument shifted by $(k-1)/2$ as compared to the standard definition, so that the functional equations relates $L_{f_i}(s)$ and $L_{f_i}(1-s)$. Under the assumption of GRH for L-functions $L_{f_i}(s)$, he proves that asymptotically their zeroes become uniformly distributed on the critical line (Theorem 1.1). This beautiful result is obtained with the help of explicit formulae and other analytic methods.

In another joint paper with Lebacque [13], for any curve X over a finite field $\mathbb{F}_q$, the authors give a lower and an upper bound for the class number h of X, that is, for the number of points on the Jacobian of X over $\mathbb{F}_q$: $h_{\min}(N) \leqslant h \leqslant h_{\max}(N)$ (Corollary 2.5). The numbers $h_{\min}(N)$ and $h_{\max}(N)$ depend on a natural parameter N, which can be chosen arbitrary, and are expressed explicitly in terms of the numbers of points on X over the fields $\mathbb{F}_{q^f}$, where $1 \leqslant f \leqslant N$. The proof of the bounds is based on an explicit formula for zeta functions of curves found by Serre, in which the authors make a suitable choice of the test function, and is also based on fine bounds on terms in the explicit formula. It is shown in the paper that for asymptotically exact families of curves $\{X_i\}$, the sequences $\log h_{\min}(N, X_i)/g_i$, $\log h(X_i)/g_i$ and $\log h_{\max}(N, X_i)/g_i$ have the same limits when $N, i \to \infty$ (Remark 2.8). Moreover, the authors present many of examples of curves from various asymptotically optimal towers, for which the above lower bound $h_{\min}(N)$ with suitable N is far better than many other known lower bounds for the class number found by other researchers (Section 3).

In [14], also joint with Lebacque, the authors consider primitive cusp forms f and Dirichlet characters χ and study distribution of values of the function $\mathscr{L}(f \otimes \chi, s)$, which denotes either the logarithm, or the logarithmic derivative of the L-function $L(f \otimes \chi, s)$. More precisely, for a function $\Phi(w)$ of a complex variable from a rather wide class, the authors consider the average value $\frac{1}{m}\sum_{\chi}\Phi(\mathscr{L}(f \otimes \chi, s))$, where s is fixed and χ runs over all Dirichlet characters with a prime conductor m. Under the assumption of GRH for $L(f \otimes \chi, s)$ it is proved that when $m \to \infty$, this average value tends to $\int_{\mathbb{C}} \Phi_\sigma(w) M_\sigma(w)|dw|$, where the function $M_\sigma(w)$ is defined explicitly in terms of the form f and of the real number $\sigma = \operatorname{Re} s$ (Theorem 4.1). One can say that $M_\sigma(w)$ is the limit distribution of values of the function $\chi \mapsto \mathscr{L}(f \otimes \chi, s)$. Moreover, also under the assumption of GRH for $L(f \otimes \chi, s)$, for any quasi-character $\psi \colon \mathbb{C} \to \mathbb{C}^*$, the authors prove statements on the limit of the average values $\operatorname{Avg}_\chi \psi(\mathscr{L}(f \otimes \chi, s))$ and $\operatorname{Avg}_f^h \psi(\mathscr{L}(f \otimes \chi, s))$, where they take averages with respect to χ and f, and the limit is taken with respect to a prime conductor m of χ and a prime level N of f, respectively (Theorems 3.1 and 5.1). The average on f is taken with certain special harmonic weights.

In the joint article with Stéphane Ballet [15], using known results on intervals between primes, the authors construct asymptotically optimal towers of modular curves over a finite field, which lead to new upper bounds on the symmetric tensor rank of multiplication in certain finite fields (Propositions 7 and 10). In a wide range of examples, these bounds are better than the bounds known before.

We hope that this collection of works will be quite useful for mathematicians from various domains and will help the memory of our dear Aliosha be longlasting.

Sergey Gorchinskiy, Evgeny Smirnov, Michael Tsfasman

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Words of colleagues and friends from the website of the Faculty of Mathematics of HSE

Alexei Pirkovskii

I remember Liosha at the time he was a student at the Independent University. In 2002–2004 he attended my lectures on functional analysis and spectral theory there, and it was a real pleasure to discuss mathematical questions with him. It was striking how deep he could see things that, frankly, did not directly concern his algebraic specialization. Even among extremely bright students of the Independent University, Liosha was distinguished by his brightness.

Several years ago, already at the Faculty of Mathematics of HSE, many times I involuntary listened to Liosha's lectures on number theory (the open door of his lecture room was just opposite the open door of my office). I listened with admiring envy — the lectures were just brilliant, both from the point of view of clarity and consistency of exposition and from the point of view of oratory. Few are those able to do the same.

We lost a talented mathematician, an outstanding teacher and simply a very good man. I cannot get out of my mind the lines of the song of Yury Vizbor[1]: "The best of the best are leaving us earlier than others, it is strange..."

Ivan Cheltsov

For a long time, I heard about Liosha Zykin as a young and very talented mathematician working at HSE. But I got acquainted with him only when he started heading the Laboratory of Algebraic Geometry. From the first meeting I realized that Liosha was very bright and good. I should add that, somehow, he did everything with ease: in mathematical research, organizing summer schools in Yaroslavl, heading the laboratory, teaching, traveling, he succeeded in everything. When Liosha got his post in Tahiti I was fantastically glad for him. To be engaged in mathematical research and to head the scientific laboratory in a place with an ideal climate and European civilization. One can only dream about it. Naturally, it was sad that he was leaving. But then it turned out that Liosha preserved his ties with Moscow and continued to take part in Moscow mathematical life. Everything was very favourable for him.

[1] Russian singer and songwriter

Young, gifted, active, athletic, a beautiful wife, living both in Tahiti and in Moscow. 32 years. The whole life is ahead. The news of his death was a shock. It is very difficult to accept.

Vladlen Timorin

It is an irreplaceable loss for mathematics and for all of us. Liosha was a talented scholar. His achievements were highly appreciated at the international level. The bright star died out though it could illuminate much more in science. We are mourning this loss and present our condolences to relatives and friends of Alexey and Tatiana. We will always remember them.

Ian Marshall

How awful! This is terribly sad news indeed, and a major loss for our Moscow community.

Fedor Bogomolov

The death of Liosha Zykin is a terrible tragedy. It is a great loss for all who knew him and worked with him. I got acquainted with him when he was a student attending the summer school in Goettingen. It is painful to write about him in the past tense, a young, full of strength and energy person who has already achieved a lot with bright prospects ahead of him. Many opportunities were opened to him as a talented scientist and organizer. Last years, I worked a lot with him on matters concerning the Laboratory. It is worth saying that in spite of his youth and lack of experience he proved to be an excellent organizer and leader. A gentle and tactful person by nature, he could be efficient and persistent. Liosha was an absolutely reliable person to whom one could always turn to in tackling serious matters. He continued to participate actively in the work of the Laboratory even when he left for Tahiti, particularly assuming the most difficult part of work relating to the functioning of the yearly school in Yaroslavl.

He was gifted in various fields, knew a lot and was interested in many things which lay far apart from mathematics. That was why it was always interesting to talk to him. Alexey was a remarkable person. My memory of him and sadness of his loss will stay forever in my heart.

Vladimir Zhgoon

It is hard to believe in the sudden and bitter loss of our dear friend Liosha Zykin. He was an outstanding mathematician, excellent lecturer and very responsible teacher who was always ready to help his students. I also remember him as a very active tourist. When visiting practically any country he knew always where to go and what to see. He was a connoisseur of a delicious cuisine and good wine. As a traveler he visited many interesting and extraordinary places. He enjoyed talking about them and did it vividly.

Rest in peace, dear Liosha!

Pavel Solomatin

I still cannot believe in what happened. Alexey Ivanovich was not only my main teacher. He was a true friend, the one who could serve as an example and who could be always consulted in one's hour of need. Why I did not quit my studies at the Faculty of Mathematics in moments of despair? Why did I start to study number theory? Why L-functions? Why curves over finite fields? The answer is simple. Thanks to Liosha. We worked together starting from his first days at the department in 2009 and continued our cooperation even after both of us left Moscow. I asked him quite recently whether he would be willing to act as one of the referees for my thesis and he naturally gave his consent. I planned to write him a letter one of these days to seek advice on what else to do in my life, but it was too late. I always thought that I would have time to write with him more than one article, to spend time together somewhere in the mountains in France with a bottle of good wine, to demonstrate that the efforts he put in us were not futile. But I was not in time to do it, and that makes my pain even greater.

Sergey Gorchinskiy

I think that for all of us this tragic news is hard to take in. I knew Liosha for many years. Starting from his first year at the University, we were studying together a lot, attended the same seminars. He was an outstanding person. Probably, in Moscow there is almost no one anymore who understands algebraic and arithmetic geometry as well as Liosha did, and at the same time is quite familiar with advanced methods of the analytic number theory and clearly understands applications to coding theory and cryptography. This amazing combination was a characteristic feature of all his mathematical creativity.

As for his early results, I would like to recall a substantial strengthening of the classical Brauer—Siegel theorem on the behavior of the regulator and the discriminant for a wide range of families of number fields. Together with P. Lebacque he obtained a new fine estimate of the logarithmic derivative of zeta functions of global fields that they used for a wide generalization of the explicit formula in the Brauer—Siegel theorem itself. Liosha was intensively developing the theory of asymptotic zeta functions, in particular, its version for families of modular forms. He developed foundations of the general asymptotic theory of varieties over finite fields. Together with his coauthors Liosha answered an important question of the great mathematician J.-P. Serre how to determine whether a principally polarized abelian threefold is a Jacobian of a curve over a non-algebraically closed field of characteristic zero. Serre evaluated this result as being quite high.

Liosha was a very good and reliable friend. One could always turn to him for help. It is amazing how he was equally attentive and patient to all people who surrounded him.

He was very sensible to life and was a great connoisseur of its beauty. His fine taste was apparent far beyond mathematics: in art, in communicating with people, in languages, in his hobbies. Liosha was an extremely well-educated person, for example, he had a deep knowledge of literature, especially French. Liosha was a fantastic organizer, incredibly combining gentleness and tactfulness in communicating with people with the ability to carry to completion all he did.

May his memory live forever. Let us remember him as often as possible, in this way we could help him now.

Alexey Rudakov

How awful! Liosha Zykin, young, energetic, and cheerful. He had been always that way in corridors and rooms of our building in Vavilova street so recently. And he is not with us any longer. I mourn him deeply and present condolences to his relatives and dear ones, to his friends and colleagues!

Sergey Loktev

The thirty-two years of Alexey's life were bright and interesting. I was lucky to know him well both as a colleague (we taught calculus together to students of the Faculty of Mathematics who graduated in 2014) and as a friend (we did some rock climbing together, travelled to Vorgol and El Chorro).

I would like to single out two of his qualities that come to my mind. First, his elegance that attracted your attention from the first moments of your contact with him. He was strikingly tactful and at the same time sincere, open to everything new and unusual, he respected life in all its expressions. Second, and his friends were very much aware of this, he was really fearless. I mean that he might experience fear but it never governed his thoughts and actions.

Our exchange of letters stopped with his words that “the rainy season is coming to its end, life is becoming even better”. I did not have time to reply to him…

May his memory live forever in our hearts!

Armen Sergeev

I am stunned by the news that Alexey Zykin and his wife died when diving.

I had the opportunity to take part in his life, though quite formally, when I was a supervisor in his post-graduate course. He was a pleasant person and certainly a very gifted mathematician.

It is a great pity to lose people so young, when even without that, we have a constant shortage of young talented people.

I express my heartfelt condolences to the parents of Alexey and of his wife.

Valentina Kiritchenko

I remember Liosha as a schoolboy. In the ninth grade it was already obvious that he was a mathematician. For three years, twice a week, he would submit to me his exercises in calculus. In Liosha’s class the schoolboys were assigned to university students (that is, you could submit your exercises only to “your own” student), and I was assigned to Liosha as his supervising student.

Already at that time, Liosha was distinguished by his fundamental approach. He never tried to solve a problem in the shortest and easiest way to get rid of it as soon as possible and to get a good grade. On the contrary, in each exercise he saw the possibility of investigating a more general matter. For example, submitting a traditional exercise on the length of a circle, he started by examining the general definition of the length of a rectifiable curve. Liosha always began by writing down his answers in a notebook, and with his approach he needed quite a number of notebooks. Often before submitting another bone-rattling solution from another notebook Liosha said: “For greater certainty let us also prove this lemma”. He was not afraid of any difficulties.

Some years later, Liosha became the youngest employee of the Faculty of Mathematics of HSE. We were colleagues both at the IPPI and at HSE. But when I think about Liosha, first there appears in my memory his image of nearly 20 years back, the image of the ninth-year schoolboy-mathematician who is not afraid of the arduous paths.

Алексей Иванович Зыкин (1984—2017)

13 июня 2019 года исполнилось бы 35 лет Алексею Зыкину — замечательному математику и преподавателю, нашему дорогому другу и коллеге.

Алеша родился в 1984 году в Москве. Его родители не были математиками: отец, Иван Семенович — доктор юридических наук, профессор, один из ведущих специалистов в России в области международного частного права, гражданского, торгового права, арбитража; мать, Юлия Ивановна — экономист в области внешней торговли.

В 14 лет Алеша поступил в математический класс в знаменитой московской школе № 57. Этот класс учили Рафаил Калманович Гордин и Петр Валентинович Сергеев, а уроки «матанализа» (так в 57-й называется углубленный курс математики) помогали вести, в частности, Александр Кузнецов и Валентина Кириченко, оба — высококлассные математики-исследователи. Именно тогда, еще в школе, Алеша заинтересовался теорией чисел и, в частности, проштудировал классический учебник Айерленда и Роузена.

В 2000 году, еще одиннадцатиклассником, Алеша поступил в Независимый московский университет, а в 2001-м, окончив школу — на мехмат МГУ, как и большинство его одноклассников. Примерно тогда же, на первом курсе, он начал заниматься научной работой под руководством Михаила Анатольевича Цфасмана. Его первая статья, «Теоремы Брауэра—Зигеля и Цфасмана—Влэдуца для почти нормальных расширений глобальных полей», была опубликована, когда он был еще на четвертом курсе. В течение всех дальнейших лет учебы в университете и в аспирантуре основным Алешиным руководителем продолжал быть М. А. Цфасман, существеннейшим образом повлиявший как на содержание математических исследований Алеши, так и на сам их стиль.

Блестяще окончив Независимый университет в 2005 году и мехмат в 2006-м и поступив в аспирантуру в Математический институт им. В. А. Стеклова РАН, Алеша получил стипендию правительства Французской республики, благодаря которой он мог проводить шесть месяцев в году во Франции. Там он учился в аспирантуре Университета Экс-Марсель II в Люмини (пригород Марселя); его соруководителем с французской стороны был Сергей Георгиевич Влэдуц. В июне 2009 года он защитил кандидатскую диссертацию во Франции, а затем в октябре 2010 года — в России.

В наши дни случаи, чтобы молодой математик после защиты диссертации сразу нашел бы постоянную академическую работу, достаточно редки, обычно этому предшествуют несколько лет постдока. Однако Алеше это удалось: в 2009 году он получил позицию доцента на появившемся незадолго до того факультете математики Высшей школы экономики. Несколько лет он оставался самым молодым сотрудником этого факультета — и одним из самых активных: он читал обязательные и специальные курсы в Вышке и в Независимом университете, организовывал многочисленные семинары и руководил первыми научными работами студентов, причем не обязательно по теории чисел. Так, например, один из студентов матфака, Дмитрий Грищенко, написал под Алешиным руководством работу о математике оригами, которая впоследствии была опубликована в сборнике «Математическое просвещение». В 2011/13 гг. по результатам студенческого голосования Алеше ежегодно присуждалось звание лучшего преподавателя ВШЭ.

Алеша был талантливым организатором. Среди наиболее важных Алешиных достижений можно назвать организацию летней школы «Алгебра и геометрия» в Ярославле, которая впервые прошла в 2011 году и с тех пор проходит ежегодно. Эта школа рассчитана на студентов 3—5 курсов и аспирантов; тем самым она является идейным продолжением знаменитой летней школы для младшекурсников «Современная математика» в Дубне, проходящей с 2001 года (и в работе которой Алеша, кстати, тоже неоднократно участвовал: сначала как слушатель, потом как преподаватель). Также в течение 2012/13 гг. Алеша был заведующим Лабораторией алгебраической геометрии и ее приложений в Высшей школе экономики.

В 2014 году — в неполные 30 лет, опять-таки исключительный случай! — Алеша получил постоянную профессорскую позицию во Франции. Вернее, в самом дальнем ее уголке, в Университете Французской Полинезии, на острове Таити. Однако, даже находясь на другом конце земного шара, он продолжал принимать самое активное участие в московской математической жизни: поддерживал контакты с московскими коллегами, руководил студентами и неизменно приезжал на ярославскую школу. Незадолго до гибели он возглавил исследовательскую группу (аналог кафедры) по алгебраической геометрии и ее приложениям к теории информации в своем университете.

Однако математикой Алешины интересы далеко не ограничивались: он был разносторонне образованным и эрудированным человеком, хорошо знал литературу, свободно владел английским и французским языками, любил путешествовать — кажется, он объездил почти весь мир, от Гималаев до Килиманджаро, — увлекался спортом, в частности, скалолазанием и дайвингом...

22 апреля 2017 года Алеша Зыкин, его жена Таня Макарова и инструктор по дайвингу Жиль Дэме трагически погибли при погружении в подводную пещеру на атолле Аэ на островах Туамоту, во Французской Полинезии.

Алеша оставил обширное научное наследие, успев написать 15 опубликованных работ (две из которых вышли уже посмертно). За свои результаты он был удостоен ряда престижных наград: премии Московского математического общества (2011), премии фонда «Династия» (2010) и других. Его работы относятся в основном к асимптотической теории глобальных полей и арифметических многообразий. Этот активно развивающийся раздел современной математики находится на стыке аналитической теории чисел, алгебраической теории чисел и алгебраической геометрии. Его основы были заложены в работах М. А. Цфасмана и С. Г. Влэдуца.

Опишем в целом суть вопросов в данной области. Важнейшими объектами математических исследований являются системы полиномиальных уравнений с целыми коэффициентами или, обобщая, арифметические многообразия. Отметим, что одномерный случай — это глобальные поля. С каждым арифметическим многообразием связана некоторая комплексно-аналитическая функция от одной переменной, называемая его дзета-функцией. Существует глубокая связь между аналитическими свойствами дзета-функции и свойствами исходного арифметического многообразия. Каждое новое утверждение, подтверждающее эту связь, представляет собой значительный интерес. Если рассматривать бесконечное семейство глобальных полей или арифметических многообразий, то оказывается, что в широком ряде случаев пределы их дзета-функций обладают многими замечательными свойствами, что имеет важные следствия, касающиеся самих арифметических многообразиях в семействе. При этом важным условием на семейство является его асимптотическая точность, что, впрочем, не является сильным ограничением, поскольку любое бесконечное семейство содержит асимптотически точное подсемейство. Именно эти вопросы и изучаются в асимптотической теории глобальных полей и, более обще, в асимптотической теории арифметических многообразий. Дополнительный интерес данных исследований заключается в многочисленных приложениях возникающих конструкций к теории кодирования и криптографии. Зыкин внес фундаментальный вклад в развитие этих областей.

Все статьи Зыкина написаны прекрасным языком, отличаются кристальной четкостью и ясностью изложения. Ключевые моменты рассуждений подробно объяснены. Во многих статьях в конце приведен список дальнейших открытых вопросов. Помимо своей высокой научной ценности, статьи Зыкина являются замечательным введением в асимптотическую теорию глобальных полей и арифметических многообразий для широкого круга математиков.

Опишем статьи Зыкина чуть подробнее.

В работе [1] доказано усиление классической теоремы Брауэра—Зигеля. Более точно, для башни числовых полей $\{K_i\}$ рассматривается предел отношения $\log(h_{K_i}R_{K_i})/g_{K_i}$, где $g_{K_i}=\log\sqrt{|D_{K_i}|}$, а h_{K_i}, R_{K_i} и D_{K_i} обозначают число классов, регулятор и дискриминант поля K_i, соответственно. Классическая теорема Брауэра—Зигеля утверждает, что если поля K_i нормальны над $\mathbb{Q}$ или выполняется обобщенная гипотеза Римана (ОГР), то при некоторых ограничениях данный предел равен 1. В статье Зыкина получен, при тех же ограничениях (в терминологии Цфасмана—Влэдуца, так называемый асимптотически плохой случай) аналогичный результат для почти нормальных полей (теорема 1). Там же замечено, что результаты Цфасмана—Влэдуца, обобщающие теорему Брауэра—Зигеля, дают аналогичное утверждение и для асимптотически хороших башен. Кроме того, в предположении ОГР построены новые примеры башен с предельным отношением Брауэра—Зигеля более близким к оценке снизу, чем известные ранее (теорема 4).

В статье [2] содержится обзор результатов Цфасмана, Влэдуца, Зыкина и Лебака о семействах глобальных полей, обзор результатов Кунявского—Цфасмана и Андри—Пачеко о семействах эллиптических кривых над функциональными и числовыми полями, соответственно. Кроме того, для асимптотически точного семейства многообразий $\{X_i\}$ размерности d над конечным полем доказана теорема о предельном поведении вычетов в точке $s=d$ дзета-функций $\zeta_{X_i}(s)$ (теорема 3.2).

Далее, в работе [3] рассматривается эллиптическая кривая E над функциональным полем K, башня расширений $\{K_i\}$ поля K, и изучается предельное поведение L-функций $L_{E_i}(s)$ эллиптических кривых $E_i = E \times_K K_i$ над K_i. Сформулировано утверждение о предельном поведении ведущих коэффициентов при разложении функций $L_{E_i}(s)$ в ряд Тэйлора в точке $s=1$ (теорема 2, п. 3). Помимо рассмотрения предельного поведения вычетов и ведущих коэффициентов дзета- и L-функций, естественно также рассмотреть предельное поведение самих данных функций, как функций комплексного переменного, определенных на подходящей области. Зыкин формулирует в [3] утверждение о предельном поведении функций $\log L_{E_i}(s)$ в области $\operatorname{Re} s > 1$ (теорема 2, п. 2). Полные доказательства обоих утверждений приведены в работе [11].

В заметке [4] кратко сформулированы результаты, подробное изложение которых содержится в статье [5]. Основные результаты статьи [5] заключаются в следующем. Для асимпототически точных семейств числовых полей $\{K_i\}$ исследуется предельное поведение логарифмов дзета-функций $\log \zeta_{K_i}(s)$. При этом во всех результатах предполгается выполненой ОГР. Доказывается, что в области $\operatorname{Re} s > 1/2$ предел функций $\log((s-1)\zeta_{K_i}(s))/g_{K_i}$ равен логарифму предельной дзета-функции $\log \zeta_{\{K_i\}}(s)$ семейства $\{K_i\}$, введенной ранее Влэдуцом и Цфасманом (теорема 2). Это дает концептуальное объяснение обобщенной теоремы Брауэра—Зигеля, а также в качестве приложение дает результат о предельном поведении констант Эйлера—Кронекера, являющийся аналогом для числовых полей результата Ихары в функциональном случае (следствие 1). Кроме того, дается нетривиальная верхняя оценка на предел логарифма ведущих коэффициентов при разложении функций $\zeta_{K_i}(s)$ в точке $s=1/2$ (теорема 3). Доказательства используют оценки на логарифмические производные дзета-функций в критической полосе, а также результаты о предельном распределении нулей дзета-функций на критической прямой в семействах числовых полей.

В короткой статье [6] анонсированы результаты из работы [7]. В работе [7], совместной с Ж. Лашо и К. Ритценталером, получен ответ на важный вопрос великого математика Ж.-П. Серра, как определить, является ли главнополяризованное абелево трехмерное многообразие (A, a) над произвольным полем $k \subset \mathbb{C}$ якобианом кривой над k.

С этой целью рассматривается некоторый арифметический инвариант $\chi_{18}(A, a, \omega) \in k$, где ω — базис в пространстве регулярных 1-форм на A. Этот инвариант выражается через аналитическую модулярную форму Зигеля $\widetilde{\chi}_{18}$ и позволяет различать абелевы трехмерные многообразия, изоморфные над квадратичным расширением основного поля. В сочетании с одним результатом Серра это дает ответ на исходный вопрос (теорема 1.3.3). Существенным образом, он сводится к тому, что (A, a) является якобианом негиперэллиптической кривой тогда и только тогда $\chi_{18}(A, a, \omega)$ является ненулевым квадратом в k. Кроме того, в статье дается новое, простое и красивое доказательство классической формулы Клейна, тесно связанной с приведенным выше вопросом и заключающейся в равенстве $\mathrm{Disc}(F)^2 = \chi_{18}(A, a, \omega)$, для гладкого однородного многочлена $F(x_1, x_2, x_3)$ степени 4 и якобиана A соответствующей плоской квартики с естественными поляризацией a и базисом ω из 1-форм (теорема 2.2.3).

В заметке [8] кратко приведены результаты из работы [9]. В работе [9], совместной с Ф. Лебаком, детально исследуется предельное поведение логарифмических производных $Z_K(s) = \zeta'_K(s)/\zeta_K(s)$ дзета-функций $\zeta_K(s)$ глобальных полей K. При этом во всех результатах, относящихся к числовым полям, предполагается выполненной ОГР. Заметим, что поскольку дзета-функция $\zeta_K(s)$ задается бесконечным произведением, функция $Z_K(s)$ задается бесконечным рядом. Сначала в статье [9] доказывается тонкая явная оценка на остаточный член в выражении как бесконечного ряда функции $Z_K(s)$ в области $\mathrm{Re}\, s > 1/2$ (теоремы 1.1 и 1.2). При доказательстве этого результата авторами продемонстрировано невероятно виртуозное владение сложнейшей аналитической техникой и явными формулами Вейля. В частности, данная оценка приводят к новому доказательству основных неравенств в асимптотической теории глобальных полей (замечания 2.2 и 3.2). Затем полученная оценка применяется к логарифмической производной $Z_{\{K_i\}}(s)$ предельной дзета-функции $\zeta_{\{K_i\}}(s)$ асимптотически точного семейства глобальных полей $\{K_i\}$. А именно, получается явная оценка на остаточный член в выражении как бесконечного ряда функции $Z_{\{K_i\}}(s)$ в области $\mathrm{Re}\, s > 1/2$ (следствие 1.3). Кроме того, авторами найдена явная оценка на остаточный член в выражении как бесконечного ряда значения $Z_{\{K_i\}}(1/2)$ (теорема 1.4). Наконец, из этого выведена явная оценка на остаточный член в выражении как бесконечного ряда значения $\log \zeta_{\{K_i\}}(1)$

(следствие 1.5). Последняя оценка является значительным усилением классической теоремы Брауэра—Зигеля.

Совместная с Лебаком статья [10] является обзором асимптотической теории и является прекрасным введением в нее. Сначала в статье излагаются основы данной теории, заложенные Цфасманом и Влэдуцем: предельные инварианты, понятие асимптотически точного семейства, основное неравенство, являющееся далеким обобщением одновременно оценок Одлыжко—Серра и неравенства Дринфельда—Влэдуца (параграф 2). Затем обсуждаются обобщения теоремы Брауэра- Зигеля, принадлежащие Цфасману, Влэдуцу и авторам статьи, предельное поведение дзета-функций и их нулей, а также удивительные связи этих тем с предельной дзета-функцией (параграф 3). Далее приводятся примеры башен функциональных полей, являющихся асимптотически оптимальными, т.е. достигающих оценку из основного неравенства (параграф 4). Такие башни соответствуют итерированным накрытиям кривых над конечным полем, имеющим в пределе наибольшее возможное число точек. В обзоре описан широкий ряд примеров асимптотически оптимальных башен, построенных Ихарой, Цфасманом, Влэдуцом, Цинком, Элкисом, Гарсией и Штихтенотом. Кроме того, обсуждается многомерное обобщение асимптотической теории глобальных полей (параграф 5). А именно, формулируются результаты Лашо—Цфасмана, обобщающие основное неравенство на случай асимптотически точных семейств многообразий над конечным полем. Также приводятся гипотетические обобщения теоремы Брауэра—Зигеля на случай семейств абелевых многообразий над функциональным полем, принадлежащие Кунявскому—Цфасману и Андри—Пачеко. Наконец, кратко объясняется формализм абстрактных L-функций над конечным полем, подробно изложенный в следующей статье.

В статье [11] заложены фундаментальные основы общей асимптотической теории многообразий над конечными и над функциональными полями. Основные утверждения из асимптотической теории функциональных полей обобщаются на случай бесконечных семейств абстрактных дзета- и L-функций над конечным полем. С этой целью Зыкин тщательно анализирует, какие именно общие арифметические свойства дзета-функций кривых приводят к данным основным утверждениям. Оказывается, что, по большому счету, достаточно потребовать аналог утверждения о модулях собственных значений Фробениуса, а также аналог неотрицательности числа точек или его ослабление, заключающееся в асимптотической очень точности семейств (определение 3.10). Развивая явные формулы в таком абстрактном контексте (параграф 2.2), автор выводит из них множество нетривиальных результатов. Отметим теорему о предельном распределение нулей (теорема 4.1), вариант обобщенной теоремы Брауэра—Зигеля о предельном поведении дзета-функций (теоремы 5.5 и 5.9) и вариант основного неравенства (теоремы 6.1 и 6.6). В качестве приложения получено утверждение о предельном поведении высших постоянных Эйлера—Кронекера для семейств функциональных полей, усиливающее известные ранее результаты Ихары (следствие 5.16), а также найдено новое доказательство основного неравенства (замечание 6.5). Кроме того, все общие результаты прекрасно проиллюстрированы приложениями к семействам эллиптических кривых над функциональными полями (следствие 4.9, теорема 5.27).

В короткой и элегантной статье [12] исследуются семейства примитивных параболических форм f_i уровня N_i и веса k_i, для которых число $N_i k_i^2$ стремится к бесконечности. Для каждой формы f_i рассматривается ее L-функция $L_{f_i}(s)$, аргумент которой по сравнению со стандартным подходом сдвинут на $(k-1)/2$ так, чтобы функциональное уравнение связывало функции $L_{f_i}(s)$ и $L_{f_i}(1-s)$. В предположении ОГР для L-функций $L_{f_i}(s)$ доказывается, что в пределе их нули распределены равномерно на критической прямой (теорема 1.1). Этот красивый результат получается с помощью явных формул и других аналитических методов.

В работе [13], совместной с Лебаком, для произвольной кривой X над конечным полем $\mathbb{F}_q$ приводятся нижняя и верхняя оценки на число классов h кривой X, т.е. на число точек на якобиане кривой X над $\mathbb{F}_q$: $h_{\min}(N) \leqslant h \leqslant h_{\max}(N)$ (следствие 2.5). При этом числа $h_{\min}(N)$ и $h_{\max}(N)$ зависят от натурального параметра N, который можно выбирать произвольным образом, и выражаются явно в терминах чисел точек кривой X над полями $\mathbb{F}_{q^f}$, где $1 \leqslant f \leqslant N$. Для доказательства оценок используется явная формула для дзета-функций кривых, предложенная Серром, в которой авторами выбирается подходящая тестовая функция, а также находятся тонкие оценки на члены, входящие в явную формулу. В статье показано, что для асимптотически точных семейств кривых $\{X_i\}$ последовательности $\log h_{\min}(N, X_i)/g_i$, $\log h(X_i)/g_i$ и $\log h_{\max}(N, X_i)/g_i$ имеют одинаковые пределы при $N, i \to \infty$ (замечание 2.8). Более того, приведен ряд примеров кривых, возникающих из различных асимптотически оптимальных башен, для которых приведенная в статье нижняя оценка $h_{\min}(N)$ при подходящем N оказывается существенно сильнее некоторых известных ранее нижних оценок на число классов, найденных другими исследователями (параграф 3).

В статье [14], также написаной совместно с Лебаком, рассматриваются примитивные параболические формы f, характеры Дирихле χ и изучается распределение значений функции $\mathscr{L}(f\otimes\chi, s)$, обозначающей логарифм или логарифмическую производную L-функции $L(f\otimes\chi, s)$. Более точно, для функции комплексного переменного $\Phi(w)$ из достаточно широкого класса рассматривается среднее значение $\frac{1}{m}\sum\limits_{\chi}\Phi(\mathscr{L}(f\otimes\chi, s))$ при фиксированном s, где χ пробегает все характеры Дирихле с простым кондуктором m. В предположении ОГР для $L(f\otimes\chi, s)$ доказывается, что при $m\to\infty$ данное среднее значение стремится к $\int\limits_{\mathbb{C}}\Phi_\sigma(w)M_\sigma(w)|dw|$, где функция $M_\sigma(w)$ определена явным образом по форме f и по вещественному числу $\sigma=\operatorname{Re} s$ (теорема 4.1). Можно сказать, что $M_\sigma(w)$ является предельным распределением значений функции $\chi\mapsto\mathscr{L}(f\otimes\chi, s)$. Кроме того, также в предположении ОГР для $L(f\otimes\chi, s)$, в статье для произвольного квази-характера $\psi\colon\mathbb{C}\to\mathbb{C}^*$ доказываются утверждения о пределе средних значений $\operatorname{Avg}_\chi\psi(\mathscr{L}(f\otimes\chi, s))$ и $\operatorname{Avg}_f^h\psi(\mathscr{L}(f\otimes\chi, s))$, где средние значения берутся по χ и f, а предел берется по простому кондуктору m характера χ и по простому уровню N формы f, соответственно (теоремы 3.1 и 5.1). При этом среднее значение по f берется с некоторыми специальными гармоническими весами.

В работе [15], совместной с С. Балле, при помощи известных результатов об интервалах между простыми числами, строятся асимптотически оптимальные семейства модулярных кривых над конечным полем, которые позволяют найти новые оценки сверху на симметрический тензорный ранг умножения в некоторых конечных полях (предложения 7 и 10). Данные оценки оказываются в ряде случаев лучше известных ранее оценок.

Мы надеемся, что данный сборник статей будет весьма полезным для математиков из различных областей, а также поможет продлить память о дорогом Алеше.

С. О. Горчинский, Е. Ю. Смирнов, М. А. Цфасман

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Слова коллег и друзей с сайта факультета математики ВШЭ

А. Ю. Пирковский

Я помню Лешу студентом Независимого университета. В 2002—2004 годах он слушал там мои лекции по функциональному анализу и спектральной теории, и общаться с ним на математические темы было настоящим удовольствием. Поражало то, насколько глубоко ему удавалось разобраться в вещах, которые, в общем-то, не относились напрямую к его алгебраической специализации. Даже на общем ярком фоне студентов Независимого Леша выделялся еще ярче. А несколько лет назад, уже на матфаке Вышки, я неоднократно оказывался невольным слушателем Лешиных лекций по теории чисел (открытая дверь его аудитории находилась в точности напротив открытой двери моего офиса). Я слушал и завидовал белой завистью — лекции были просто великолепны, как с точки зрения четкости и последовательности изложения, так и с точки зрения ораторского искусства. Мало кто так умеет.

Мы потеряли талантливого математика, выдающегося преподавателя и просто очень хорошего человека. Не выходят из головы строки из песни Юрия Визбора: «Лучшие ребята из ребят раньше всех уходят, это странно...»

И. А. Чельцов

Я узнал о Леше Зыкине давно как о молодом и очень талантливом математике в Вышке. А познакомился с ним только когда он стал руководить лабораторией Алгебраической Геометрии. И с первой встречи стало понятно что Леша очень светлый и хороший человек. При этом он как-то легко все делал: занимался математикой, организовывал летние школы в Ярославле, руководил лабораторией, преподавал, путешествовал. И у него все получалось. Когда Леша получил позицию на Таити, я был безумно рад за него. Заниматься математикой и руководить научной лабораторией в месте с идеальным климатом и европейской цивилизацией. О таком можно только мечтать. Конечно было грустно, что он уезжает. Но потом оказалось, что Леша сохранил связь с Москвой и продолжал активно участвовать в Московской математической жизни. В общем все сложилось очень хорошо. Молодой, талантливый, активный, спортивный, жена красавица, живет на Таити и в Москве. 32 года. Вся жизнь впереди. Новость о его смерти шокировала. Очень трудно это принять.

В. А. Тиморин

Невосполнимая утрата для математики и для всех нас. Леша был талантливым ученым, его достижения получили высокое международное признание, но это было только начало. Яркая звезда погасла, хотя могла бы еще многое осветить в науке. Скорбим и соболезнуем родственникам и друзьям Алексея и Татьяны. Мы будем помнить о них.

И. Маршалл

How awful! This is terribly sad news indeed, and a major loss for our Moscow community.

Ф. А. Богомолов

Произошла страшная трагедия. Погиб Леша Зыкин.

Это огромная потеря для всех, кто его знал и работал вместе с ним. Я знал его еще студентом, с тех пор как он приехал летом на школу в Геттинген. Больно писать о нем в прошедшем времени, о молодом, полном сил и энергии человеке, который уже многого достиг и перед которым открывались еще большие перспективы. С его талантом ученого и организатора перед ним были открыты многие пути. Последние годы я много с ним работал по делам Лаборатории и хочу отметить, что несмотря на молодость и отсутствие опыта он проявил себя прекрасным организатором и руководителем. Будучи по природе мягким и тактичным он умел быть четким и настойчивым. Леша был абсолютно надежным и на него всегда можно было положиться в решении серьезных вопросов. Даже после своего отъезда на Таити он продолжал активно участвовать в работе Лаборатории и, в частности, взял на себя самую трудную часть работы по организации ежегодной Ярославской школы.

Он был талантлив в разных областях, много знал и интересовался многими вещами далекими от математики. Поэтому было всегда интересно с ним разговаривать. Леша был замечательным человеком. Память о нем и горечь от этой утраты навсегда останется в моем сердце.

В. С. Жгун

Трудно поверить в столь неожиданную и горькую потерю нашего дорогого друга Леши Зыкина. Он был замечательный математик, отличный лектор и очень ответственный педагог, готовый всегда прийти на помощь своим студентам. А еще я помню его как очень активного туриста. Находясь в почти любой стране, он всегда знал, куда можно поехать, что посмотреть. Он был знатоком изысканной кухни и хорошего вина, а в его копилке путешественника множество интересных и необычных мест, о которых он с удовольствием и красочно рассказывал…

Пусть земля тебе будет пухом, дорогой Леша!

П. Соломатин

До сих пор не могу поверить. Алексей Иванович был не просто моим главным учителем. Он был мне настоящим другом, одним из тех на кого хотелось равняться и у кого всегда можно было спросить совета в трудную минуту. Почему я не бросил учебу на матфаке в моменты отчаяния? Почему занялся теорией чисел? Почему L-функции? Почему кривые над конечными полями? Ответ простой — благодаря Леше. Мы работали вместе с ним начиная с его первых дней на факультете в 2009 году и продолжали работать даже после того как оба уехали из Москвы. Буквально недавно я спросил не хочет ли он быть одним из моих научных оппонентов на защите диссертации и он естественно согласился. И вот на днях я собирался написать ему очередное письмо, хотел спросить совета как двигаться по жизни дальше, но как оказалось, не успел. Я всегда думал, что успею…успею написать с ним не одну статью, успею посидеть где-нибудь в горах во Франции с бутылкой хорошего вина, успею показать, что те усилия которые он в нас вкладывал были не напрасны. Но не успел. И от этого как-то вдвойне больно.

С. О. Горчинский

Думаю, у всех нас эта трагическая новость никак не помещается в голове.

Я знал Лешу много лет, начиная с его первого курса университета, мы многому учились с ним вместе, ходили на одни семинары. Это был ярчайший человек.

Наверное, в Москве больше почти нет людей, которые также как Леша прекрасно понимают алгебраическую и арифметическую геометрию, в то же время свободно владеют продвинутыми методами аналитической теории чисел, и ясно понимают приложения к кодированию и криптографии. Это удивительное сочетание проходило через все математическое творчество Леши. Из его ранних результатов мне хочется вспомнить существенное усиление классической теоремы Брауэра—Зигеля о поведении регулятора и дискриминанта для широкого ряда последовательностей числовых полей. Вместе с Ф. Лебаком им была получена новая тонкая оценка на логарифмическую производную дзета-функций глобальных полей, которая была ими применена для далекого обобщения явной формулы в самой теореме Брауэра—Зигеля. Леша интенсивно развивал теорию асимптотических дзета-функций, расширяя ее на случай семейств модулярных форм. Он заложил фундаментальные основы общей асимптотической теории дзета-функций многообразий над конечными полями. Совместно с соавторами, Леша ответил на важный вопрос великого математика Ж.-П. Серра о том, как определить, является ли якобианом кривой главнополяризованное абелево трехмерное многообразие над не алгебраически замкнутым полем нулевой характеристики. Серр был высокого мнения об этом результате. Леша был очень хорошим надежным товарищем. К нему всегда можно было обратиться за помощью. Удивительно, как одинаково внимательно и терпеливо он относился ко всем окружающим его людям.

Он очень чутко ощущал жизнь, был высоким ценителем прекрасного в ней. Его тонкий вкус проявлялся далеко за пределами математики, в искусстве, общении с людьми, языках, увлечениях. Леша был на редкость широко образованным человеком, например, хочется вспомнить его глубокое знание литературы, особенно французской. Леша был фантастическим организатором, невероятно сочетая мягкость и тактичность в общении с людьми со способностью доводить все дела до конца в совершенном виде.

Пусть память о Леше будет вечной. Давайте чаще его вспоминать, так мы можем помочь ему теперь.

А. Н. Рудаков

Как ужасно! Алеша Зыкин, молодой, энергичный и веселый — таким он был всегда в коридорах и комнатах нашего здания на Вавилова так недавно — и его уже нет на этой Земле. Глубоко скорблю и соболезную родным и близким, друзьям и коллегам!

С. А. Локтев

Алексей за 32 года прожил яркую и интересную жизнь. Мне посчастливилось хорошо узнать его и как коллегу (мы вместе учили математическому анализу студентов матфака выпуска 2014 года), и как друга (мы вместе занимались скалолазанием, вместе ездили на Воргол и в Эль-Чорро).

Я хотел бы отметить два его качества, каким он запомнился. Во-первых, это — элегантность, которая бросалась в глаза с первых минут общения с ним. Он был потрясающе тактичен, и в то же время искренен, открыт ко всему новому и необычному, ценил жизнь во всех ее проявлениях.

Во-вторых, и это хорошо знали его друзья, он был по-настоящему бесстрашен. Не в том смысле, что не испытывал страх, а в том, что страх никогда не обуславливал его мысли и действия.

Наша переписка оборвалась на том, что «сезон дождей заканчивается — жизнь становится еще лучше». Я не успел ему ответить…

Пусть светлая память о Леше пребывает в наших сердцах!

А. Г. Сергеев

Я ошеломлен новостью о гибели Алексея Зыкина и его жены во время дайвинга.

Мне довелось принять, хотя и формальное участие в его судьбе, будучи его руководителем в аспирантуре. Это был приятный и безусловно очень способный математик.

Очень жаль терять таких молодых людей, когда и так мы постоянно ощущаем их нехватку.

Присоединяюсь к соболезнованиям родителям Алексея и его жены.

В. А. Кириченко

Я помню Лешу школьником. В девятом классе уже было очевидно, что он математик. На протяжении трех лет два раза в неделю Леша сдавал мне задачи из листков по кматанализуњ. В Лешином классе школьников прикрепляли к студентам (то есть, сдавать задачи можно было только «своему» студенту), а меня как раз назначили студентом для Леши.

Уже тогда Лешу отличал фундаментальный подход. Он никогда не стремился решить задачу самым дешевым и коротким способом, лишь бы скорей сдать и получить плюсик. Наоборот, в каждой задаче он видел возможность детально разобраться в более общем вопросе. Например, сдавая традиционную для листков задачу о длине окружности, Леша сначала разобрал общее определение длины спрямляемой кривой. Свои решения Леша всегда сначала записывал в тетрадь, и тетрадей при его подходе требовалось довольно много. Часто перед тем как рассказать очередное зубодробительное утверждение из очередной тетради Леша говорил «Для пущей ясности докажем еще такую лемму». Его не пугали никакие сложности.

Прошли годы, Леша из школьника стал самым молодым сотрудником матфака Вышки. Мы с ним были коллегами в ИППИ и в Вышке. Однако когда я думаю о Леше, в моей памяти в первую очередь возникает его образ почти 20-летней давности, образ девятиклассника-математика, который не боится идти трудным путем.

The Brauer—Siegel and Tsfasman—Vlăduţ theorems for almost normal extensions of number fields

To my teacher M. A. Tsfasman
on the occasion of his 50th birthday

Abstract. The classical Brauer—Siegel theorem states that if k runs through the sequence of normal extensions of $\mathbb{Q}$ such that $\frac{n_k}{\log|D_k|} \to 0$, then $\frac{\log h_k R_k}{\log\sqrt{|D_k|}} \to 1$. First, in this paper we obtain the generalization of the Brauer—Siegel and Tsfasman—Vlăduţ theorems to the case of almost normal number fields. Second, using the approach of Hajir and Maire, we construct several new examples concerning the Brauer—Siegel ratio in asymptotically good towers of number fields. These examples give smaller values of the Brauer—Siegel ratio than those given by Tsfasman and Vlăduţ.

1. Introduction

Let K be an algebraic number field of degree $n_K = [K:\mathbb{Q}]$ and discriminant D_K. We define the genus of K as $g_K = \log\sqrt{D_K}$. By h_K we denote the class-number of K, R_K denotes its regulator. We call a sequence $\{K_i\}$ of number fields a family if K_i is non-isomorphic to K_j for $i \neq j$. A family is called a tower if also $K_i \subset K_{i+1}$ for any i. For a family of number fields we consider the limit

$$\mathrm{BS}(\mathscr{K}) := \lim_{i\to\infty} \frac{\log h_{K_i} R_{K_i}}{g_{K_i}}.$$

The classical Brauer—Siegel theorem, proved by Brauer (see [1]), states that for a family $\mathscr{K} = \{K_i\}$ we have $\mathrm{BS}(\mathscr{K}) = 1$ if the family satisfies two conditions:

(i) $\lim\limits_{i\to\infty} \frac{n_{K_i}}{g_{K_i}} = 0$;

(ii) either the generalized Riemann hypothesis (GRH) holds, or all the fields K_i are normal over $\mathbb{Q}$.

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We call a number field almost normal if there exists a finite tower of number fields $\mathbb{Q}=K_0\subset K_1\subset\ldots\subset K_m=K$ such that all the extensions K_i/K_{i-1} are normal. Weakening the condition (ii), we prove the following generalization of the classical Brauer—Siegel theorem to the case of almost normal number fields:

Theorem 1. *Let $\mathscr{K}=\{K_i\}$ be a family of almost normal number fields for which $n_{K_i}/g_{K_i}\to 0$ as $i\to\infty$. Then we have* $\mathrm{BS}(\mathscr{K})=1$.

It was shown by M. A. Tsfasman and S. G. Vlăduţ that, taking in account non-archimedian places, one may generalize the Brauer—Siegel theorem to the case of extensions where the condition (i) does not hold.

For a prime power q we set

$$N_q(K_i) := |\{v\in P(K_i)\colon \operatorname{Norm}(v)=q\}|,$$

where $P(K_i)$ is the set of non-archimedian places of K_i. We also put $N_{\mathbb{R}}(K_i)=r_1(K_i)$ and $N_{\mathbb{C}}(K_i)=r_2(K_i)$, where r_1 and r_2 stand for the number of real and (pairs of) complex embeddings.

We consider the set $A=\{\mathbb{R},\mathbb{C};2,3,4,5,7,8,9,\ldots\}$ of all prime powers plus two auxiliary symbols $\mathbb{R}$ and $\mathbb{C}$ as the set of indices. A family $\mathscr{K}=\{K_i\}$ is called asymptotically exact if and only if for any $\alpha\in A$ the following limit exists:

$$\phi_\alpha=\phi_\alpha(\mathscr{K}) := \lim_{i\to\infty}\frac{N_\alpha(K_i)}{g_{K_i}}.$$

We call an asymptotically exact family $\mathscr{K}$ asymptotically good (respectively, bad) if there exists $\alpha\in A$ with $\phi_\alpha>0$ (respectively, $\phi_\alpha=0$ for any $\alpha\in A$). The condition on a family to be asymptotically bad is, in the number field case, obviously equivalent to the condition (i) in the classical Brauer—Siegel theorem. For an asymptotically good tower of number fields the following generalization of the Brauer—Siegel theorem was proved in [11]:

Theorem 2 (Tsfasman—Vlăduţ Theorem, see [11, Theorem 7.3])**.** *Assume that for an asymptotically good tower $\mathscr{K}$ fields any of the following conditions is satisfied*:

- *GRH holds*;
- *all the fields K_i are almost normal over $\mathbb{Q}$.*

Then the limit $\mathrm{BS}(\mathscr{K})=\lim\limits_{i\to\infty}\dfrac{\log h_{K_i}R_{K_i}}{g_{K_i}}$ *exists and we have*:

$$\mathrm{BS}(\mathscr{K}) = 1+\sum_q \phi_q\log\frac{q}{q-1}-\phi_{\mathbb{R}}\log 2-\phi_{\mathbb{C}}\log 2\pi, \tag{1}$$

the sum beeing taken over all prime powers q.

For an asymptotically bad tower of number fields we have $\phi_{\mathbb{R}} = 0$ and $\phi_{\mathbb{C}} = 0$ as well as $\phi_q = 0$ for all prime powers q, so the right hand side of the formula (1) equals to one. We also notice that the condition on a family to be asymptotically bad is equivalent to $\lim\limits_{i\to\infty} \frac{n_{K_i}}{g_{K_i}} = 0$. So, combining our Theorem 1 with Theorem 2 we get the following corollary:

Corollary 3. *For any tower $\mathcal{K} = \{K_i\}$, $K_1 \subset K_2 \subset \ldots$ of almost normal number fields the limit* $\mathrm{BS}(\mathcal{K})$ *exists and we have*:

$$\mathrm{BS}(\mathcal{K}) = \lim_{i\to\infty} \frac{\log(h_i R_i)}{g_i} = 1 + \sum_q \phi_q \log \frac{q}{q-1} - \phi_{\mathbb{R}} \log 2 - \phi_{\mathbb{C}} \log 2\pi,$$

the sum beeing taken over all prime powers q.

In [11] bounds on the ratio $\mathrm{BS}(\mathcal{K})$ were given, together with examples showing that the value of $\mathrm{BS}(\mathcal{K})$ may be different from 1. We corrected some of these erroneous bounds and managed to precise a few of the estimates in the examples. Also, using the infinite tamely ramified towers, found by Hajir and Maire (see [3]), we get (under GRH) new examples, both totaly complex and totally real, with the values of $\mathrm{BS}(\mathcal{K})$ smaller than those of totally real and totally complex examples of [11]. The result is as follows:

Theorem 4. 1. *Let $k = \mathbb{Q}(\xi)$, where ξ is a root of*

$$f(x) = x^6 + x^4 - 4x^3 - 7x^2 - x + 1,$$

$$K = k(\sqrt{\xi^5 - 467\xi^4 + 994\xi^3 - 3360\xi^2 - 2314\xi + 961}).$$

Then K is totally complex and has an infinite tamely ramified 2-tower $\mathcal{K}$, for which, under GRH, we have:

$$\mathrm{BS}_{\mathrm{lower}} \leqslant \mathrm{BS}(\mathcal{K}) \leqslant \mathrm{BS}_{\mathrm{upper}},$$

where $\mathrm{BS}_{\mathrm{lower}} \approx 0.56498\ldots$, $\mathrm{BS}_{\mathrm{upper}} \approx 0.59748\ldots$.

2. *Let $k = \mathbb{Q}(\xi)$, where ξ is a root of*

$$f(x) = x^6 - x^5 - 10x^4 + 4x^3 + 29x^2 + 3x - 13,$$

$$K = k(\sqrt{-2993\xi^5 + 7230\xi^4 + 18937\xi^3 - 38788\xi^2 - 32096\xi + 44590}).$$

Then K is totally real and has an infinite tamely ramified 2-tower $\mathcal{K}$, for which, under GRH, we have:

$$\mathrm{BS}_{\mathrm{lower}} \leqslant \mathrm{BS}(\mathcal{K}) \leqslant \mathrm{BS}_{\mathrm{upper}},$$

where $\mathrm{BS}_{\mathrm{lower}} \approx 0.79144\ldots$, $\mathrm{BS}_{\mathrm{upper}} \approx 0.81209\ldots$.

However, unconditionally (without GRH), the estimates for totally complex fields that may be obtained using the methods developed by Tsfasman and Vlăduţ lead to slightly worse results, than those already known from [11]. This is due to a rather large number of prime ideals of small norm in the field K. For the same reasons the upper bounds for the Brauer—Siegel ratio for other fields constructed in [3] are too high, though the lower bounds are still good enough.

Finally we present the table (the ameliorated version of the table of [11]), where all the bounds and estimates are given together:

		lower bound	lower example	upper example	upper bound
GRH	all fields	0.5165	0.5649—0.5975	1.0602—1.0798	1.0938
	totally real	0.7419	0.7914—0.8121	1.0602—1.0798	1.0938
	totally complex	0.5165	0.5649—0.5975	1.0482—1.0653	1.0764
Unconditional	all fields	0.4087	0.5939—0.6208	1.0602—1.1133	1.1588
	totally real	0.6625	0.8009—0.9081	1.0602—1.1133	1.1588
	totally complex	0.4087	0.5939—0.6208	1.0482—1.1026	1.1310

2. Proof of Theorem 1

Let $\zeta_K(s)$ be the Dedekind zeta function of the number field K and $\varkappa_K$ its residue at $s=1$. By w_K we denote the number of roots of unity in K, and by r_1, r_2 the number of real and complex places of K respectively. We have the following residue formula (see [4, Ch. VIII, § 3]):

$$\varkappa = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K \sqrt{D_K}}.$$

Since

$$\sqrt{w_K/2} \leqslant \varphi(w_K) = [\mathbb{Q}(\zeta_{w_K}) : \mathbb{Q}] \leqslant [K : \mathbb{Q}] = n_K,$$

we note that $w_K \leqslant 2n_K^2$ so $\log w_{K_j}/g_{K_j} \to 0$. Thus, it is enough to prove that $\log \varkappa_{K_j}/\log D_{K_j} \to 0$.

As for the upper bound we have

Theorem 5 (See [5, Theorem 1]). *Let K be a number field of degree $n \geqslant 2$. Then,*

$$\varkappa_K \leqslant \left(\frac{e \log D_K}{2(n-1)}\right)^{n-1}. \tag{2}$$

Moreover, $1/2 \leqslant \rho < 1$ and $\zeta_K(\rho)=0$ imply

$$\varkappa_K \leqslant (1-\rho)\left(\frac{e \log D_K}{2n}\right)^{n}. \tag{3}$$

Using the estimate (2) we get (even without the assumption of almost normality) the "easy inequality":

$$\frac{\log \varkappa_{K_j}}{\log D_{K_j}} \leqslant \frac{n_j - 1}{\log D_{K_j}} \Bigl(\log \frac{e}{2} + \log \frac{\log D_{K_j}}{n_j - 1} \Bigr) \to 0.$$

As for the lower bound the business is much more tricky and we will proceed to the proof after giving a few preliminary statements.

Let K be a number field other than $\mathbb{Q}$. A real number ρ is called an *exceptional zero* of $\zeta_K(s)$ if $\zeta_K(\rho) = 0$ and

$$1 - (4 \log D_K)^{-1} \leqslant \rho < 1;$$

an exceptional zero ρ of $\zeta_K(s)$ is called its *Siegel zero* if

$$1 - (16 \log D_K)^{-1} \leqslant \rho < 1.$$

Our proof will be based on the following fundamental property of Siegel zeroes proved by Stark:

Theorem 6 (see [10, Lemma 10]). *Let K be an almost normal number field, and let ρ be a Siegel zero of $\zeta_K(s)$. Then there exists a quadratic subfield k of K such that $\zeta_k(\rho) = 0$.*

The next estimate is also due to Stark:

Theorem 7 (See [10, Lemma 4] or [6, Theorem 1]). *Let K be a number field and let ρ be the exceptional zero of $\zeta_K(s)$ if it exists and $\rho = 1 - (4 \log D_K)^{-1}$ otherwise. Then there is an absolute constant $c < 1$ (effectively computable) such that*

$$\varkappa_K > c(1 - \rho). \tag{4}$$

Our proof of Theorem 1 will be similar to the proof of the classical Brauer—Siegel theorem given in [7]. We will use the Brauer—Siegel result for quadratic fields, a simple proof of which is given in [2]. There are two cases to consider.

1. First, assume that $\zeta_{K_j}(s)$ has no Siegel zero. From (4) we deduce that

$$\varkappa_{K_j} > c(1 - \rho) \geqslant c\Bigl(1 - \Bigl(1 - \frac{1}{16 \log D_{K_j}}\Bigr)\Bigr) = \frac{c}{16 \log D_{K_j}}. \tag{5}$$

2. Second, assume that there exists a Siegel zero ρ of $\zeta_{K_j}(s)$. From Theorem 6 we see that there exists a quadratic subfield k_j of K_j such that $\zeta_{k_j}(\rho) = 0$. Applying (3) and (4) we obtain:

$$\varkappa_{K_j} = \frac{\varkappa_{K_j}}{\varkappa_{k_j}} \varkappa_{k_j} \geqslant \frac{c(1 - \rho)}{(1 - \rho)\Bigl(\dfrac{e \log D_{k_j}}{4}\Bigr)^2} \varkappa_{k_j} = \frac{16c}{e^2 \log^2 D_{k_j}} \varkappa_{k_j}. \tag{6}$$

If the number of fields K_j for which the second case holds is finite, then, using the fact that $\log D_{K_j} \to \infty$, we get the desired lower estimate from (5).

Otherwise, we note that for a number field there exists at most one exceptional zero (See [10, Lemma 3], so, applying this statement to the fields k_j, we get that only finitely many of them may be isomorphic to each other and so $D_{k_j} \to \infty$ as $j \to \infty$. Thus we may use the Brauer—Siegel result for quadratic fields:

$$\frac{\log \varkappa_{k_j}}{\log D_{K_j}} \leqslant \frac{\log \varkappa_{k_j}}{\log D_{k_j}} \to 0.$$

Finally from (6), we get:

$$\frac{\log \varkappa_{K_j}}{\log D_{K_j}} \geqslant \frac{16c}{e^2 \log D_{K_j}} - 2\frac{\log\log D_{k_j}}{\log D_{K_j}} + \frac{\log \varkappa_{k_j}}{\log D_{K_j}} \to 0.$$

This concludes the proof. □

Remark 8. Our proof of Theorem 1 is explicit and effective if all the fields in the family $\mathcal{K}$ contain no quadratic subfield and thus the corresponding zeta function does not have Siegel zeroes.

3. Proof of Theorem 4

First we recall briefly some constructions related to class field towers. Let us fix a prime number ℓ. For a finitely generated pro-ℓ group G, we let $d(G) = \dim_{\mathbb{F}_\ell} H^1(G, \mathbb{F}_\ell)$ be its generator rank. Let T be a finite set of ideals of a number field K such that no prime in T is a divisor of ℓ. We denote by K_T the maximal ℓ-extension of K unramified outside T, $G_T = \mathrm{Gal}(K_T/K)$. We let

$$\theta_{K,T} = \begin{cases} 1, & \text{if } T \neq \varnothing \text{ and } K \text{ contains a primitive } \ell\text{th root of unity;} \\ 0, & \text{otherwise.} \end{cases}$$

Then we have (see [9, Theorems 1 and 5]):

Theorem 9. *If* $d(G_T) \geqslant 2 + 2\sqrt{r_1(K) + r_2(K) + \theta_{K,T}}$, *then* K_T *is infinite.*

To estimate $d(G_T)$ we use the following theorem

Theorem 10 (See [8, Section 2]). *Let* K/k *be a finite Galois extension,* $r_1 = r_1(k)$, $r_2 = r_2(k)$, ρ *be the number of real places of* k, *ramified in* K, t *be the number of primes in* k, *ramified in* K. *We set* $\delta_\ell = 1$, *if* k *contains a primitive root of degree* ℓ *of unity and* $\delta_\ell = 0$ *otherwise. Then* *we have*:

$$d(G_T) \geqslant d(G_\varnothing) \geqslant t - r_1 - r_2 + \rho - \delta_\ell.$$

The number field arithmetic behind the construction of our theorem 4 was mainly carried out with the help of the computer package PARI. However, we would like to present our examples in the way suitable for non-computer check. We give here the proof of the first part of our theorem, as the proof of the second part is very much similar and may be carried out simply by repeating all the steps of the proof given here.

We let $k=\mathbb{Q}(\xi)$, where ξ is a root of $f(x)=x^6+x^4-4x^3-7x^2-x+1$. Then k is a field of signature (4, 1) and discriminant $d_f=d_k=-23\cdot 35509$. Its ring of integers is $\mathcal{O}_k=\mathbb{Z}[\xi]$ and its class number is equal to 1. The principle ideal of norm $7\cdot 13\cdot 19^2\cdot 23^2\cdot 29\cdot 31$ generated by

$$\eta = 671\xi^5 - 467\xi^4 + 994\xi^3 - 3360\xi^2 - 2314\xi + 961$$

factors into eight different prime ideals of $\mathcal{O}_k$. In fact, one may see that $\eta = \pi_7\pi_{13}\pi_{19}\pi'_{19}\pi_{23}\pi'_{23}\pi_{29}\pi_{31}$, where

$$\begin{aligned}
\pi_7 &= -9\xi^5 + 6\xi^4 - 13\xi^3 + 44\xi^2 + 31\xi - 12,\\
\pi_{13} &= -7\xi^5 + 5\xi^4 - 11\xi^3 + 36\xi^2 + 23\xi - 9,\\
\pi_{19} &= 5\xi^5 - 4\xi^4 + 8\xi^3 - 26\xi^2 - 15\xi + 6,\\
\pi'_{19} &= 5\xi^5 - 3\xi^4 + 7\xi^3 - 24\xi^2 - 20\xi + 6,\\
\pi_{23} &= -5\xi^5 + 4\xi^4 - 8\xi^3 + 26\xi^2 + 15\xi - 9,\\
\pi'_{23} &= 6\xi^5 - 4\xi^4 + 9\xi^3 - 30\xi^2 - 22\xi + 6,\\
\pi_{29} &= 11\xi^5 - 8\xi^4 + 17\xi^3 - 56\xi^2 - 35\xi + 16,\\
\pi_{31} &= 7\xi^5 - 5\xi^4 + 11\xi^3 - 36\xi^2 - 22\xi + 7.
\end{aligned}$$

$K=k(\sqrt{\eta})$ is a totally complex field of degree 12 over $\mathbb{Q}$ with the relative discriminant $\mathcal{D}_{K/k}$ equal to (η) as $\eta=\beta^2+4\gamma$, where $\beta=\xi^5+\xi^4+\xi^3+1$, $\gamma=-173\xi^5+112\xi^4-270\xi^3+815\xi^2+576\xi-237$. From this we see that $d_K = 7\cdot 13\cdot 19^2\cdot 23^2\cdot 29\cdot 31\cdot 23^2\cdot 35509^2$. From Theorem 10 we deduce that

$$d(G_\varnothing) \geqslant t - r_1(k) - r_2(k) + \rho - 1 = 8 - 4 - 1 + 4 - 1 = 6.$$

The right hand side of the inequality from Theorem 9 is equal to $2+2\sqrt{6}\approx$ $\approx 6.8989 < 7$, so it is enough to show that $d(G_T) > d(G_\varnothing)$, and to do this it is enough to construct a set of prime ideals T and an extension of K, ramified exactly at T.

Let $\pi_3 = -6\xi^5 + 4\xi^4 - 9\xi^3 + 30\xi^2 + 21\xi - 7$ be the generator of a prime ideal of norm 3 in $\mathcal{O}_k$ and T be the set consisting of one prime ideal of $\mathcal{O}_K$ over $\pi_3\mathcal{O}_k$. We see that

$$\pi_3\pi_{19} = 11\xi^5 - 8\xi^4 + 17\xi^3 - 56\xi^2 - 35\xi + 14 = \rho^2 + 4\sigma,$$

where $\rho = \xi^5 + \xi^3 + \xi^2 + 1$, $\sigma = 2\xi^5 - 8\xi^4 - 14\xi^3 - 28\xi^2 - 9\xi + 5$, so $k(\sqrt{\pi_3\pi_{19}})/k$ is ramified exactly at π_3 and π_{19}. But π_{19} already ramifies in K that is why $K(\sqrt{\pi_3\pi_{19}})/K$ is ramified exactly at T. Thus we have showed that $d(G_T) \geqslant 7$ and K_T/K is indeed infinite.

To complete our proof we need a few more results.

Theorem 11 (GRH Basic Inequality, see [11, Theorem 3.1]). *For an asymptotitically exact family of number fields under GRH one has*:

$$\sum_q \frac{\phi_q \log q}{\sqrt{q} - 1} + \phi_{\mathbb{R}}\Big(\log 2\sqrt{2\pi} + \frac{\pi}{4} + \frac{\gamma}{2}\Big) + \phi_{\mathbb{C}}(\log 8\pi + \gamma) \leqslant 1, \qquad (7)$$

the sum beeing taken over all prime powers q.

Theorem 12 (See [3, Theorem 1]). *Let K be a number field of degree n over $\mathbb{Q}$, such that K_T is infinite and assume that $K_T = \bigcup_{i=1}^{\infty} K_i$. Then*

$$\lim_{i\to\infty} \frac{g_i}{n_i} \leqslant \frac{g_K}{n_K} + \frac{\sum\limits_{\mathfrak{p}\in T} \log(N_{K/\mathbb{Q}}\mathfrak{p})}{2n_K}.$$

For our previously constructed field K the genus is equal to $g_K \approx \approx 25.3490\ldots$ From Theorem 12 we easily see that $\phi_{\mathbb{R}} = 0$ and

$$\frac{12}{2g_K + 2\log 3} \leqslant \phi_{\mathbb{C}} \leqslant \frac{12}{2g_K},$$

i. e., $0.23669 < \phi_{\mathbb{C}} < 0.22687$. The lower bound for $\mathrm{BS}(K_T)$ is clearly equal to

$$\mathrm{BS}_{\mathrm{lower}} = 1 - \phi_{\mathbb{R}} \log 2 - \phi_{\mathbb{C}} \log(2\pi) \leqslant 0.56498\ldots.$$

Knowing the decomposition in K of small primes of $\mathbb{Q}$, we may now apply the linear programming approach to get the upper bound for $\mathrm{BS}(K_T)$. This is done using the explicit formula (1) for the Brauer—Siegel ratio along with the basic inequality (7) and the inequality

$$\sum_{m=1}^{\infty} m\phi_{p^m} \leqslant \phi_{\mathbb{R}} + 2\phi_{\mathbb{C}},$$

taken as the restrictions. This was done using the PARI package. As the calculations are rather cumbersome we will give here only the final result: $\mathrm{BS}_{\mathrm{upper}} \approx 0.59748\ldots$, and the bound is attained for $\phi_7 = \phi_9 = \phi_{13} = 0.03944\ldots$, $\phi_{19} = 0.01002\ldots$ $\square$

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On the generalizations of the Brauer—Siegel theorem

Abstract. The classical Brauer—Siegel theorem states that if k runs through the sequence of normal extensions of $\mathbb{Q}$ such that $\frac{n_k}{\log|D_k|} \to 0$, then $\frac{\log(h_k R_k)}{\log\sqrt{|D_k|}} \to 1$. In this paper we give a survey of various generalizations of this result including some recent developements in the study of the Brauer—Siegel ratio in the case of higher dimensional varieties over global fields. We also present a proof of a higher dimensional version of the Brauer—Siegel theorem dealing with the study of the asymptotic properties of the residue at $s = d$ of the zeta function in a family of varieties over finite fields.

1. Introduction

Let K be an algebraic number field of degree $n_K = [K : \mathbb{Q}]$ and discriminant D_K. We define the genus of K as $g_K = \log\sqrt{D_K}$. By h_K we denote the class-number of K, R_K denotes its regulator. We call a sequence $\{K_i\}$ of number fields a family if K_i is non-isomorphic to K_j for $i \neq j$. A family is called a tower if also $K_i \subset K_{i+1}$ for any i. For a family of number fields we consider the limit

$$\mathrm{BS}(\mathcal{K}) := \lim_{i\to\infty} \frac{\log(h_{K_i} R_{K_i})}{g_{K_i}}.$$

The classical Brauer—Siegel theorem, proved by Brauer (see [3]) can be stated as follows:

Theorem 1.1 (Brauer—Siegel). *For a family $\mathcal{K} = \{K_i\}$ we have*

$$\mathrm{BS}(\mathcal{K}) := \lim_{i\to\infty} \frac{\log(h_{K_i} R_{K_i})}{g_{K_i}} = 1$$

if the family satisfies two conditions:

(i) $\lim\limits_{i\to\infty} \frac{n_{K_i}}{g_{K_i}} = 0;$

(ii) *either the generalized Riemann hypothesis (GRH) holds, or all the fields K_i are normal over $\mathbb{Q}$.*

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The initial motivation for the Brauer—Siegel theorem can be traced back to a conjecture of Gauss:

Conjecture 1.2 (Gauss). *There are only 9 imaginary quadratic fields with class number equal to one, namely those having their discriminants equal to* -3, -4, -7, -8, -11, -19, -43, -67, -163.

The first result towards this conjecture was proven by Heilbronn in [11]. He proved that $h_K \to \infty$ as $D_K \to -\infty$. Moreover, together with Linfoot [12] he was able to verify that Gauss' list was complete with the exception of at most one discriminant. However, this "at most one" part was completely ineffective. The initial question of Gauss was settled independently by Heegner [10], Stark [28] and Baker [1] (initially the paper by Heegner was not acknowledged as giving the complete proof). We refer to [35] for a more thorough discussion of the history of the Gauss class number problem.

A natural question was to find out what happens with the class number in the case of arbitrary number fields. Here the situation is more complicated. In particular a new invariant comes into play: the regulator of number fields, which is very difficult to separate from the class number in asymtotic considerations (in particular, for this reason the other conjecture of Gauss on the infinitude of real quadratic fields having class number one is still unproven). A major step in this direction was made by Siegel [27] who was able to prove Theorem 1.3 in the case of quadratic fields. He was followed by Brauer [3] who actually proved what we call the classical Brauer—Siegel theorem.

Ever since a lot of different aspects of the problem have been studied. For example, the major difficulty in applying the Brauer—Siegel theorem to the class number problem is its ineffectiveness. Thus many attempts to obtain good explicit bounds on $h_K R_K$ were undertaken. In particular we should mention the important paper of Stark [29] giving an explicit version of the Brauer—Siegel theorem in the case when the field contains no quadratic subfields. See also some more recent papers by Louboutin [21], [22] where better explicit bounds are proven in certain cases. Even stronger effective results were needed to solve (at least in the normal case) the class-number-one problem for CM fields, see [15], [25], [2].

In another direction, assuming the generalized Riemann hypothesis (GRH) one can obtain more precise bounds on the class number then those given by the Brauer—Siegel theorem. For example in the case of quadratic fields we have $h_K \ll D_K^{1/2}(\log\log D_K / \log D_K)$. In particular they are known to be optimal in many cases (see [5], [6], [4]).

A full survey of the problems stemming from the study of the Brauer—Siegel type questions definitely lies beyond the scope of this article. Our goal is more modest. Here we survey the results that generalize the classical Brauer—Siegel theorem. In § 2 the case of families of number fields violating one (or both) of the conditions (i) and (ii) of theorem 1.3 is discussed. In particular we introduce the notion of Tsfasman—Vlăduţ invariants of global fields that allow to express the Brauer—Siegel limit in general. In § 3 we survey the known results and conjectures about the Brauer—Siegel type statements in the higher dimensional situation. Finally, in the last § 4 we prove a Brauer—Siegel type result (Theorem 3.2) for families of varieties over finite fields. This theorem expresses the asymptotic properties of the residue at $s=d$ of the zeta function of smooth projective varieties over finite fields via the asymptotics of the number of $\mathbb{F}_{q^m}$-points on them.

2. The case of global fields: Tsfasman—Vlăduţ approach

A natural question is whether one can weaken the conditions (i) and (ii) of Theorem 1.3. The first condition seems to be the most restrictive one. Tsfasman and Vlăduţ were able to deal with it first in the function field case [31], [32] and then in the number field case [33] (which was as usual more difficult, especially from the analytical point of view). It turned out that one has to take in account non-archimedian place to be able to treat the general situation. Let us introduce the necessary notation in the number field case (for the function field case see § 3).

For a prime power q we set

$$\Phi_q(K_i) := |\{v \in P(K_i) : \mathrm{Norm}(v) = q\}|,$$

where $P(K_i)$ is the set of non-archimedian places of K_i. Taking in account the archimedian places we also put $\Phi_{\mathbb{R}}(K_i)=r_1(K_i)$ and $\Phi_{\mathbb{C}}(K_i)=r_2(K_i)$, where r_1 and r_2 stand for the number of real and (pairs of) complex embeddings.

We consider the set $A=\{\mathbb{R}, \mathbb{C}; 2, 3, 4, 5, 7, 8, 9, \ldots\}$ of all prime powers plus two auxiliary symbols $\mathbb{R}$ and $\mathbb{C}$ as the set of indices.

Definition 2.1. A family $\mathscr{K}=\{K_i\}$ is called asymptotically exact if and only if for any $\alpha \in A$ the following limit exists:

$$\phi_\alpha = \phi_\alpha(\mathscr{K}) := \lim_{i\to\infty} \frac{\Phi_\alpha(K_i)}{g_{K_i}}.$$

We call an asymptotically exact family $\mathscr{K}$ asymptotically good (respectively, bad) if there exists $\alpha \in A$ with $\phi_\alpha > 0$ (respectively, $\phi_\alpha = 0$ for any $\alpha \in A$). The ϕ_α are called the Tsfasman—Vlăduţ invariants of the family $\{K_i\}$.

One knows that any family of number fields contains an asymptotically exact subfamily so the condition on a family to be asymptotically exact is not very restrictive. On the other hand, the condition of asymptotical goodness is indeed quite restrictive. It is easy to see that a family is asymptotically bad if and only if it satisfies the condition (i) of the classical Brauer—Siegel theorem. In fact, before the work of Golod and Shafarevich [9] even the existence of asymptotically good families of number fields was unclear. Up to now the only method to construct asymptotically good families in the number field case is essentially based on the ideas of Golod and Shafarevich and consists of the usage of classfield towers (quite often in a rather elaborate way). This method has the disadvantage of beeing very inexplicit and the resulting families are hard to control (ex. splitting of the ideals, ramification, etc.). In the function field case we dispose of a much wider range of constructions such as the towers coming from supersingular points on modular curves or Drinfeld modular curves ([16], [34]), the explicit iterated towers proposed by Garcia and Stichtenoth [7], [8] and of course the classfield towers as in the number field case (see [26] for the treatement of the function field case).

This partly explains why so little is known about the above set of invariants ϕ_α. Very few general results about the structure of the set of possible values of (ϕ_α) are available. For instance, we do not know whether the set $\{\alpha \mid \phi_\alpha \neq 0\}$ can be infinite for some family $\mathscr{K}$. We refer to [20] for an exposition of most of the known results on the invariants ϕ_α.

Before formulating the generalization of the Brauer—Siegel theorem proven by Tsfasman and Vlăduţ in [33] we have to give one more definition. We call a number field almost normal if there exists a finite tower of number fields $\mathbb{Q} = K_0 \subset K_1 \subset \cdots \subset K_m = K$ such that all the extensions K_i/K_{i-1} are normal.

Theorem 2.2 (Tsfasman—Vlăduţ). *Assume that for an asymptotically good tower $\mathscr{K}$ any of the following conditions is satisfied:*

- *GRH holds;*
- *all the fields K_i are almost normal over $\mathbb{Q}$.*

Then the limit $\mathrm{BS}(\mathscr{K}) = \lim\limits_{i\to\infty} \dfrac{\log(h_{K_i} R_{K_i})}{g_{K_i}}$ *exists and we have:*

$$\mathrm{BS}(\mathscr{K}) = 1 + \sum_q \phi_q \log\frac{q}{q-1} - \phi_{\mathbb{R}} \log 2 - \phi_{\mathbb{C}} \log 2\pi,$$

the sum beeing taken over all prime powers q.

We see that in the above theorem both the conditions (i) and (ii) of the classical Brauer—Siegel theorem are weakend. A natural supplement to the above theorem is the following result obtained by the author in [36]:

Theorem 2.3 (Zykin). *Let $\mathcal{K}=\{K_i\}$ be an asymptotically bad family of almost normal number fields* (*i. e. a family for which $n_{K_i}/g_{K_i}\to 0$ as $i\to\infty$*). *Then we have* $\mathrm{BS}(\mathcal{K})=1$.

One may ask if the values of the Brauer—Siegel ratio $\mathrm{BS}(\mathcal{K})$ can really be different from one. The answer is "yes". However, due to our lack of understanding of the set of possible (ϕ_α) there are only partial results. Under GRH one can prove (see [33]) the following bounds on $\mathrm{BS}(\mathcal{K})$: $0.5165\leqslant \mathrm{BS}(\mathcal{K})\leqslant 1.0938$. The existence bounds are weaker. There is an example of a (class field) tower with $0.5649\leqslant \mathrm{BS}(\mathcal{K})\leqslant 0.5975$ and another one with $1.0602\leqslant \mathrm{BS}(\mathcal{K})\leqslant 1.0938$ (see [33] and [36]). Our inability to get the exact value of $\mathrm{BS}(\mathcal{K})$ lies in the inexplicitness of the construction: as it was said before, class field towers are hard to control. A natural question is whether all the values of $\mathrm{BS}(\mathcal{K})$ between the bounds in the examples are attained. This seems difficult to prove at the moment though one may hope that some density results (i. e. the density of the values of $\mathrm{BS}(\mathcal{K})$ in a certain interval) are within reach of the current techniques.

Let us formulate yet another version of the generalized Brauer—Siegel theorem proven by Lebacque in [19]. It assumes GRH but has the advantage of beeing explicit in a certain (unfortunately rather weak) sense:

Theorem 2.4 (Lebacque). *Let $\mathcal{K}=\{K_i\}$ be an asymptotically exact family of number fields. Assume that* GRH *in true. Then the limit* $\mathrm{BS}(\mathcal{K})$ *exists, and we have:*

$$\sum_{q\leqslant x}\phi_q\log\frac{q}{q-1}-\phi_{\mathbb{R}}\log 2-\phi_{\mathbb{C}}\log 2\pi=\mathrm{BS}(\mathcal{K})+\mathcal{O}\Big(\frac{\log x}{\sqrt{x}}\Big).$$

This theorem is an easy corollary of the generalised Mertens theorem proven in [19]. We should also note that Lebacque's apporoach leads to a unified proof of Theorems 2.2 and 2.3 with or without the assumption of GRH.

3. Varieties over global fields

Once we are in the realm of higher dimensional varieties over global fields the question of finding a proper analogue of the Brauer—Siegel theorem becomes more complicated and the answers which are currently available are far from being complete. Here we have essentially three approaches: the one by the author (which leads to a fairly simple result), another one by Kunyavskii and Tsfasman and the last one by Hindry and Pacheko (which for the moment gives only plausible conjectures). We will present all of them one by one.

The proof of the classical Brauer—Siegel theorem as well as those of its generalisations discussed in the previous section passes through the residue formula. Let $\zeta_K(s)$ be the Dedekind zeta function of a number field K and $\varkappa_K$ its residue at $s=1$. By w_K we denote the number of roots of unity in K. Then we have the following classical residue formula:

$$\varkappa_K = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K \sqrt{D_K}}.$$

This formula immediately reduces the proof of the Brauer—Siegel theorem to an appropriate asymptotical estimate for $\varkappa_K$ as K varies in a family (by the way, this makes clear the connection with GRH which appears in the statement of the Brauer—Siegel theorem). So, in the higher dimensional situation we face two completely different problems:

(i) Study the asymptotic properties of a value of a certain ζ or L-function.
(ii) Find an (arithmetic or geometric) interpretation of this value.

One knows that just like in the case of global fields in the d-dimensional situation zeta function $\zeta_X(s)$ of a variety X has a pole of order one at $s=d$. Thus the first idea would be to take the residue of $\zeta_X(s)$ at $s=d$ and study its asymptotic behaviour. In this direction we can indeed obtain a result. Let us proceed more formally.

Let X be a complete non-singular absolutely irreducible projective variety of dimension d defined over a finite field $\mathbb{F}_q$ with q elements, where q is a power of p. Denote by $|X|$ the set of closed points of X. We put $X_n = X \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$ and $\overline{X} = X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$. Let Φ_{q^m} be the number of places of X having degree m, that is $\Phi_{q^m} = |\{\mathfrak{p} \in |X| \mid \deg(\mathfrak{p}) = m\}|$. Thus the number N_n of $\mathbb{F}_{q^n}$-points of the variety X_n is equal to

$$N_n = \sum_{m|n} m\Phi_{q^m}.$$

Let $b_s(X) = \dim_{\mathbb{Q}_l} H^s(\overline{X}, \mathbb{Q}_l)$ be the l-adic Betti numbers of X. We set $b(X) = \max_{i=1\ldots 2d} b_i(X)$. Recall that the zeta function of X is defined for $\mathrm{Re}(s) > d$ by the following Euler product:

$$\zeta_X(s) = \prod_{\mathfrak{p}\in|X|} \frac{1}{1-N(\mathfrak{p})^{-s}} = \prod_{m=1}^{\infty}\Big(\frac{1}{1-q^{-sm}}\Big)^{\Phi_{q^m}},$$

where $N(\mathfrak{p})=q^{-\deg \mathfrak{p}}$. It is known that $\zeta_X(s)$ has an analytic continuation to a meromorphic function on the complex plane with a pole of order one at $s=d$. Furthermore, if we set $Z(X,q^{-s})=\zeta_X(s)$ then the function $Z(X,t)$ is a rational function of $t=q^{-s}$.

Consider a family $\{X_j\}$ of complete non-singular absolutely irreducible d-dimensional projective varieties over $\mathbb{F}_q$. We assume that the families under consideration satisfy $b(X_j)\to\infty$ when $j\to\infty$. Recall (see [18]) that such a family is called asymptotically exact if the following limits exist:

$$\phi_{q^m}(\{X_j\}) = \lim_{j\to\infty} \frac{\Phi_{q^m}(X_j)}{b(X_j)}, \qquad m=1,2,\ldots$$

The invariants ϕ_{q^m} of a family $\{X_j\}$ are called the Tsfasman—Vlăduţ invariants of this family. One knows that any family of varieties contains an asymptotically exact subfamily.

Definition 3.1. We define the Brauer—Siegel ratio for an asymptotically exact family as

$$\mathrm{BS}(\{X_j\}) = \lim_{j\to\infty} \frac{\log|\varkappa(X_j)|}{b(X_j)},$$

where $\varkappa(X_j)$ is the residue of $Z(X_j,t)$ at $t=q^{-d}$.

In § 4 we prove the following generalization of the classical Brauer—Siegel theorem:

Theorem 3.2. *For an asymptotically exact family $\{X_j\}$ the limit $\mathrm{BS}(\{X_j\})$ exists and the following formula holds:*

$$\mathrm{BS}(\{X_j\}) = \sum_{m=1}^{\infty} \phi_{q^m} \log \frac{q^{md}}{q^{md}-1}. \tag{3.1}$$

However, we come across a problem when we trying to carry out the second part of the strategy sketched above. There seems to be no easy geometric interpretaion of the invariant $\varkappa(X)$ (apart from the case $d=1$ where we have a formula relating $\varkappa_X$ to the number of $\mathbb{F}_q$-points on the Jacobian of X). See however [23] for a certain cohomological interpretation of $\varkappa(X)$.

Let us now switch our attention to the two other approaches by Kunyavskii—Tsfasman and by Hindry—Pacheko. Both of them have for their starting points the famous Birch—Swinnerton-Dyer conjecture which expresses the value at $s=1$ of the L-function of an abelian variety in terms of certain arithmetic invariants related to this variety. Thus, in this case we have (at least conjecturally) an interpretation of the special value of the L-function at $s=1$. However, the situation with the asymptotic behaviour of this value is much less clear. Let us begin with the approach of Kunyavskii—Tsfasman. To simplify our notation we restrict ourselves to the case of elliptic curves and refer for the general case of abelian varieties to the original paper [17].

Let K be a global field that is either a number field or $K=\mathbb{F}_q(X)$ where X is a smooth, projective, geometrically irreducible curve over a finite field $\mathbb{F}_q$. Let E/K be an elliptic curve over K. Let $Ш := |Ш(E)|$ be the order of the Shafarevich—Tate group of E, and Δ the determinant of the Mordell—Weil lattice of E (see [30] for definitions). Note that in a certain sense $Ш$ and Δ are the analogues of the class number and of the regulator respectively. The goal of Kunyavskii and Tsfasman in [17] is to study the asymptotic behaviour of the product $Ш\cdot\Delta$ as $g\to\infty$. They are able to treat the so-called constant case:

Theorem 3.3 (Kunyavskii—Tsfasman). *Let $E=E_0\times_{\mathbb{F}_q} K$ where E_0 a fixed elliptic curve over $\mathbb{F}_q$. Let K vary in an asymptotically exact family $\{K_i\}=\{\mathbb{F}_q(X_i)\}$, and let $\phi_{q^m}=\phi_{q^m}(\{X_i\})$ be the corresponding Tsfasman—Vlăduţ invariants. Then*

$$\lim_{i\to\infty}\frac{\log_q(Ш_i\cdot\Delta_i)}{g_i}=1-\sum_{m=1}^{\infty}\phi_{q^m}\log_q\frac{N_m(E_0)}{q^m},$$

where $N_m(E_0)=|E_0(\mathbb{F}_{q^m})|$.

Note that there is no real need to assume the above mentioned Birch and Swinnerton-Dyer conjecture as it was proven by Milne [24] in the constant case. The proof of the above theorem uses this result of Milne to get an explicit formula for $Ш\cdot\Delta$ thus reducing the proof of the theorem to the study of asymptotic properties of curves over finite fields the latter ones being much better known.

Kunyavskii and Tsfasman also make a conjecture in a certain non constant case. To formulate it we have to introduce some more notation. Let E be again an arbitrary elliptic K-curve. Denote by $\mathcal{E}$ the corresponding elliptic surface (this means that there is a proper connected smooth morphism $f:\mathcal{E}\to X$ with the generic fibre E). Assume that f fits into an infinite Galois tower, i. e. into a commutative diagram of the following form:

$$\begin{tikzcd}
\mathcal{E}=\mathcal{E}_0 \arrow[d, "f"] & \mathcal{E}_1 \arrow[l] \arrow[d] & \dots \arrow[l] & \mathcal{E}_i \arrow[l] \arrow[d] & \dots \arrow[l] \\
X=X_0 & X_1 \arrow[l] & \dots \arrow[l] & X_i \arrow[l] & \dots \arrow[l]
\end{tikzcd} \tag{3.2}$$

where each lower horizontal arrow is a Galois covering. For every $v \in X$ closed point in X, let $E_v = f^{-1}(v)$. Let $\Phi_{v,i}$ denote the number of points of X_i lying above v, $\phi_v = \lim_{i\to\infty} \Phi_{v,i}/g_i$ (we suppose the limits exist). Furthermore, denote by $f_{v,i}$ the residue degree of a point of X_i lying above v (the tower being Galois, this does not depend on the point), and let $f_v = \lim_{i\to\infty} f_{v,i}$. If $f_v = \infty$, we have $\phi_v = 0$. If f_v is finite, denote by $N(E_v, f_v)$ the number of $\mathbb{F}_{q^{f_v}}$-points of E_v. Finally, let τ denote the "fudge" factor in the Birch and Swinnerton-Dyer conjecture (see [30] for its precise definition). Under this setting Kunyavskii and Tsfasman formulate the following conjecture in [17]:

Conjecture 3.4 (Kunyavskii—Tsfasman). *Assuming the Birch and Swinnerton-Dyer conjecture for elliptic curves over function fields, we have*

$$\lim_{i\to\infty} \frac{\log_q(\text{Ш}_i \cdot \Delta_i \cdot \tau_i)}{g_i} = 1 - \sum_{v\in X} \phi_v \log_q \frac{N(E_v, f_v)}{q^{f_v}}.$$

Let us finally turn our attention to the approach of Hindry and Pacheko. They treat the case in some sense "orthogonal" to that of Kunyavskii and Tsfasman. Here, contrary to the previous setting of this section, we consider the number field case as the more complete one. We refer to [14] for the function field case. As in the approach of Kunyavskii and Tsfasman we study elliptic curves over global fields. However, here the ground field K is fixed and we let vary the elliptic curve E. Denote by $h(E)$ the logarithmic height of an elliptic curve E (see [13] for the precise definition, asymptotically its properties are close to those of the conductor). Hindry in [13] formulates the following conjecture:

Conjecture 3.5 (Hindry—Pacheko). *Let E_i run through a family of pairwise non-isomorphic elliptic curves over a fixed number field K. Then*

$$\lim_{i\to\infty} \frac{\log(\text{Ш}_i \cdot \Delta_i)}{h(E_i)} = 1.$$

To motivate this conjecture, Hidry reduces it to a conjecture on the asymptotics of the special value of L-functions of elliptic curves at $s = 1$ using the conjecture of Birch and Swinnerton-Dyer as well as that of Szpiro and Frey (the latter one is equivalent to the ABC conjecture when $K = \mathbb{Q}$).

Let us finally state some open questions that arise naturally from the above discussion.

- What is the number field analogue of Theorem 3.2?

It seems not so difficult to prove the result corresponding to Theorem 3.2 in the number field case assuming GRH. Without GRH the situation looks much more challenging. In particular, one has to be able to controll the so called Siegel zeroes of zeta functions of varieties (that is real zeroes close to $s=d$) which might turn out to be a difficult problem. The conjecture 3.4 can be easily written in the number field case. However, in this situation we have even less evidence for it since Theorem 3.3 is a particular feature of the function field case.

- How can one unify the conjectures of Kunyavskii—Tsfasman and Hindry—Pacheko?

In particular it is unclear which invariant of elliptic curves should play the role of genus from the case of global fields. It would also be nice to be able to formulate some conjectures for a more general type of L-functions, such as automorphic L-functions.

- Is it possible to justify any of the above conjectures in certain particular cases? Can one prove some cases of these conjectures "on average" (in some appropriate sense)?

For now the only case at hand is the one given by Theorem 3.3.

4. The proof of the Brauer—Siegel theorem for varieties over finite fields: case $s=d$

Recall that the trace formula of Lefschetz—Grothendieck gives the following expression for N_n — the number of $\mathbb{F}_{q^n}$ points on a variety X:

$$N_n = \sum_{s=0}^{2d}(-1)^s q^{ns/2}\sum_{i=1}^{b_s}\alpha_{s,i}^n, \tag{4.1}$$

where $\{q^{s/2}\alpha_{s,i}\}$ is the set of of inverse eigenvalues of the Frobenius endomorphism acting on $H^s(\overline{X}, \mathbb{Q}_l)$. By Poincaré duality one has $b_{2d-s}=b_s$ and $\alpha_{s,i}=\alpha_{2d-s,i}$. The conjecture of Riemann—Weil proven by Deligne states that the absolute values of $\alpha_{s,i}$ are equal to 1. One also knows that $b_0=1$ and $\alpha_{0,1}=1$.

One can easily see that for $Z(X, q^{-s})=\zeta_X(s)$ we have the following power series expansion:

$$\log Z(X,t) = \sum_{n=1}^{\infty} N_n\frac{t^n}{n}. \tag{4.2}$$

Combining (4.2) and (4.1) we obtain

$$Z(X,t) = \prod_{s=0}^{2d}(-1)^{s-1}P_s(X,t), \tag{4.3}$$

where $P_s(X,t)=\prod_{i=1}^{b_i}(1-q^{s/2}\alpha_{s,i})$. Furthermore we note that $P_0(X,t)=$ $=1-t$ and $P_{2d}(X,t)=1-q^d t$.

To prove Theorem 3.2 we will need the following lemma.

Lemma 4.1. *For $c \to \infty$ we have*

$$\frac{\log|\varkappa(X_j)|}{b(X_j)} = \sum_{l=1}^{c} \frac{N_l(X_j) - q^{dl}}{l} q^{-dl} + R_c(X_j),$$

with $R_c(X_j) \to 0$ uniformly in j.

Proof of the Lemma. Using (4.3) one has

$$\frac{\log|\varkappa(X_j)|}{b(X_j)} + d\frac{\log q}{b(X_j)} = \frac{1}{b(X_j)} \sum_{s=0}^{2d-1} (-1)^{s+1} \log|P_s(X_j, q^{-d})| =$$

$$= \frac{1}{b(X_j)} \sum_{s=0}^{2d-1} (-1)^{s+1} \sum_{k=1}^{b_s(X_j)} \log(1 - q^{(s-2d)/2}\alpha_{s,i}) =$$

$$= -\frac{1}{b(X_j)} \sum_{s=0}^{2d-1} (-1)^{s+1} \sum_{k=1}^{b_s(X_j)} \sum_{l=1}^{\infty} \frac{q^{(s-2d)l/2}\alpha_{s,i}^l}{l} =$$

$$= \frac{1}{b(X_j)} \sum_{l=1}^{c} \frac{q^{-dl}}{l} \left(\sum_{s=0}^{2d} (-1)^s q^{sl/2} \sum_{k=1}^{b_s(X_j)} \alpha_{s,i}^l - q^{dl} \right) +$$

$$+ \frac{1}{b(X_j)} \sum_{s=0}^{2d-1} (-1)^s \sum_{k=1}^{b_s(X_j)} \sum_{l=c+1}^{\infty} \frac{q^{(s-2d)l/2}\alpha_{s,i}^l}{l} =$$

$$= \sum_{l=1}^{c} \frac{N_l(X_j) - q^{dl}}{l} q^{-dl} + R_c(X_j).$$

An obvious estimate gives

$$|R_c(X_j)| \leqslant \frac{\sum_{s=0}^{2d} b_s(X_j)}{b(X_j)} \sum_{l=c+1}^{\infty} \frac{q^{-l/2}}{l} \to 0$$

for $c \to \infty$ uniformly in j. □

Now let us note that

$$\frac{1}{b(X_j)} \sum_{l=1}^{c} \frac{1}{l} \leqslant \frac{2}{b(X_j)} \log c \to 0$$

when $\log c / b(X_j) \to 0$. Thus to prove the main theorem we are left to deal with the following sum:

$$\frac{1}{b(X_j)} \sum_{l=1}^{c} \frac{q^{-ld}}{l} N_l(X_j) = \frac{1}{b(X_j)} \sum_{l=1}^{c} \frac{q^{-dl}}{l} \sum_{m|l} m\Phi_{q^m} = \frac{1}{b(X_j)} \sum_{m=1}^{c} \Phi_{q^m} \sum_{k=1}^{\lfloor c/m \rfloor} \frac{q^{-mkd}}{k} =$$

$$= \frac{1}{b(X_j)} \sum_{m=1}^{c} \Phi_{q^m} \log \frac{q^{md}}{q^{md}-1} - \frac{1}{b(X_j)} \sum_{m=1}^{c} \Phi_{q^m} \sum_{\lfloor c/m \rfloor + 1}^{\infty} \frac{q^{-mkd}}{k}.$$

Let us estimate the last term:

$$\frac{1}{b(X_j)}\sum_{m=1}^{c}\Phi_{q^m}\sum_{k=\lfloor c/m\rfloor+1}^{\infty}\frac{q^{-mkd}}{k}\leqslant$$

$$\leqslant\frac{1}{b(X_j)}\sum_{m=1}^{c}\frac{N_m(X_j)q^{-md(\lfloor c/m\rfloor+1)}}{m(\lfloor c/m\rfloor+1)(1-q^{-md})}\leqslant\frac{1}{b(X_j)}\sum_{m=1}^{c}\frac{N_m(X_j)q^{-cd}}{c(1-q^{-md})}\leqslant$$

$$\leqslant\frac{1}{b(X_j)}\sum_{m=1}^{c}\Big(q^{md}+1+\sum_{s=1}^{2d-1}b_sq^{ms/2}\Big)\frac{q^{-dc}}{c(1-q^{-md})}\leqslant$$

$$\leqslant\frac{1}{b(X_j)}\Big(q^{cd}+1+\sum_{s=1}^{2d-1}b_sq^{cs/2}\Big)\frac{q^{-dc}}{(1-q^{-1})}\to 0$$

as both $b(X_j)\to\infty$ and $c\to\infty$.

Now, to finish the proof we will need an analogue of the basic inequality from [31]. In the higher dimensional case there are several versions of it. However, here the simplest one will suffice. Let us define for $i=0\dots 2d$ the following invariants:

$$\beta_i(\{X_j\})=\limsup_j\frac{b_i(X_j)}{b(X_j)}.$$

Theorem 4.2. *For an asymptotically exact family $\{X_j\}$ we have the inequality:*

$$\sum_{m=1}^{\infty}\frac{m\phi_{q^m}}{q^{(2d-1)m/2}-1}\leqslant(q^{(2d-1)/2}-1)\Big(\sum_{i\equiv 1\bmod 2}\frac{\beta_i}{q^{(i-1)/2}+1}+\sum_{i\equiv 0\bmod 2}\frac{\beta_i}{q^{(i-1)/2}-1}\Big).$$

Proof. See [18, Remark 8.8]. □

Applying this theorem together with the fact that

$$\log\frac{q^{md}}{q^{md}-1}=O\Big(\frac{1}{q^{dm}-1}\Big)=O\Big(\frac{m}{q^{(2d-1)m/2}-1}\Big)$$

when $m\to\infty$, we conclude that the series on the right hand side of (3.1) converges. Thus the difference

$$\sum_{m=1}^{\infty}\phi_{q^m}\log\frac{q^{md}}{q^{md}-1}-\frac{1}{b(X_j)}\sum_{m=1}^{c}\Phi_{q^m}\log\frac{q^{md}}{q^{md}-1}=$$

$$=\sum_{m=1}^{c}\Big(\phi_{q^m}-\frac{\Phi_{q^m}}{b(X_j)}\Big)\log\frac{q^{md}}{q^{md}-1}-\sum_{m=c+1}^{\infty}\phi_{q^m}\log\frac{q^{md}}{q^{md}-1}\to 0$$

when $c\to\infty$, $j\to\infty$ and j is large enough compared to c. This concludes the proof of Theorem 3.2.

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Brauer—Siegel theorem for families of elliptic surfaces over finite fields

The classical Brauer—Siegel theorem for number fields proved by Brauer (see [1]) claims that, if k ranges over a sequence of number fields normal over $\mathcal{Q}$ and such that $\frac{n_k}{\log|D_k|} \to 0$, then $\frac{\log(h_k R_k)}{\log\sqrt{|D_k|}} \to 1$. Here D_k, h_k, and R_k stand for the discriminant, the class number, and the regulator of the field k, respectively. This theorem was generalized by M. A. Tsfasman and S. G. Vlăduţ (see [2]) to the case in which the condition $n_k/\log|D_k| \to 0$ fails to hold (asymptotically good families of fields). Here the limit thus obtained, $\lim \log(h_k R_k)/\log\sqrt{|D_k|}$, need not be equal to 1.

The existence of a deep analogy between number fields and function fields has been well known for a long time. Here many results for function fields can be obtained in a much simpler way (for instance, analytic problems related to zeta functions disappear). The analog of the Brauer—Siegel theorem for function fields is proved in an essentially simpler way, and the normality condition (which is present in the case of number fields) turns out to be excessive.

Let $\{X_i\}$ be a family of pairwise nonisomorphic smooth absolutely irreducible projective curves over a finite field $\mathbb{F}_q$ of genus $g_i = g(X_i)$. Let $\Phi_{q^m}(X_i)$ be the number of points whose degree is exactly equal to m on the curve X_i.

Definition 1. The numbers

$$\phi_{q^m} = \lim_{i\to\infty} \frac{\Phi_{q^m}(X_i)}{g_i}$$

are said to be the *Tsfasman—Vlăduţ invariants* of the family $\{X_i\}$. If the limits ϕ_{q^m} exist, then the family is said to be *asymptotically exact*.

Let

$$Z_i(t) = \prod_{m=1}^{\infty} (1 - t^m)^{-\Phi_m(X_i)}$$

be the zeta function of the curve X_i. It has a pole of order one at the point $t = 1/q$, and we denote the residue of the function at the point by

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κ_i. As is well known, κ_i can be expressed in terms of the $\mathbb{F}_q$-points of the Jacobian of the curve X_i (an analog of the ideal class numbers in the number field case). The following theorem holds, which was proved by Tsfasman in [3].

Theorem 1. *The formula*

$$\lim_{i\to\infty} \frac{\log \kappa_i}{g_i} = 1 + \sum_{m=1}^{\infty} \phi_{q^m} \log \frac{q^m}{q^m - 1}$$

holds for any asymptotically exact family of curves X_i.

Attempts to generalize this theorem to the multidimensional case immediately lead to several results. First, for a family of algebraic varieties of dimension d over a finite field $\mathbb{F}_q$, one can study the behavior of the residue of the zeta function at the point $t = q^{-d}$. An analog of Theorem 1 in this direction was obtained in [4]. However, the geometric interpretation of the residue of the zeta function at the point $t = q^{-d}$ is less simple here.

Another approach was suggested by Hindry in [5] and by Kunyavskii and Tsfasman in [6]. In these papers, the behavior of the value of the L-function at the point $s = 1$ is studied for families of elliptic curves. The problem is of interest, because this value is related to subtle arithmetic invariants of elliptic curves by the Birch—Swinnerton-Dyer conjecture. Hindry formulates a conjecture (similar to the Brauer—Siegel theorem) in the case of a family of elliptic curves over a chosen number field. In this note, we are mainly interested in the function case, and therefore we consider the Kunyavskii—Tsfasman approach in more detail.

Let us present several preliminary definitions. Let X be a smooth projective curve over $\mathbb{F}_q$, let $K = \mathbb{F}_q(X)$ be the function field of X, let E/K be an elliptic curve, and let $f : E \to X$ be the corresponding elliptic surface. Consider the family of coverings

$$X = X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_i \leftarrow \cdots$$

and the family of elliptic surfaces $\mathcal{E}_i$ obtained by the base change,

$$\begin{array}{ccccccccc}
\mathcal{E} = \mathcal{E}_0 & \longleftarrow & \mathcal{E}_1 & \longleftarrow & \cdots & \longleftarrow & \mathcal{E}_i & \longleftarrow & \cdots \\
\big\downarrow{\scriptstyle f} & & \big\downarrow & & & & \big\downarrow & & \\
X = X_0 & \longleftarrow & X_1 & \longleftarrow & \cdots & \longleftarrow & X_i & \longleftarrow & \cdots
\end{array}$$

Let $\Phi_{v,f}(X_i)$ be the number of points of degree f on X_i lying above the point $v \in |X|$. Assume below that the limits $\phi_{v,f} = \lim\limits_{i\to\infty}(\Phi_{v,f}(X_i)/g_i)$

exist for the family $\{X_i\}$. Denote by E_i the elliptic curve which is the common fiber of the mappings $E_i \to X_i$. If v is a closed point of X, then we set $N_v = q^{\deg v}$. Let $N_{v,f}(E)$ be the number of points on the reduction $E_v = f^{-1}(v)$ of the curve E over the field $\mathbb{F}_{Nv^f}$. Write $a_{v,f}(E) = Nv^f + 1 - N_{v,f}(E)$ and

$$L_{v,f}(E,s) = \begin{cases} (1 - a_{v,f} Nv^{-fs} + Nv^{f(1-2s)})^{-1} & \text{if } E_v \text{ is nonsingular;} \\ (1 - a_{v,f} Nv^{-fs})^{-1} & \text{otherwise.} \end{cases}$$

Recall that the L-function of an elliptic curve E is defined as

$$L_E(s) = \prod_{v \in |X|} L_{v,1}(E,s).$$

We also introduce the limit L-function of the family $\{E_i/K_i\}$ by

$$L_{\{E_i/K_i\}}(s) = \prod_{v \in |X|} \prod_{f=1}^{\infty} L_{v,f}(E,s)^{\phi_{f,v}}.$$

Let r_E be the order of zero of $L_E(s)$ at the point $s=1$, and let c_E be the first nonzero coefficient in the expansion of $L_E(s)$ in the Taylor series at $s=1$. Kunyavskii and Tsfasman [6] formulate the following conjecture.

Conjecture. $\displaystyle\lim_{i\to\infty} \frac{\log|c_{E_i}|}{g_i} = -\sum_{v\in|X|} \sum_{f=1}^{\infty} \phi_{v,f} \log \frac{N_{v,f}(E)}{Nv^f}.$

A special case of this conjecture in the case of constant elliptic curves ($\mathcal{E}_i = E' \times X_i$, where $E'/\mathbb{F}_q$ is a chosen elliptic curve) is also proved in [6]. Unfortunately, the proof contains gaps. The transposition of the passage to the limit in the infinite product for the L-function and the passage to the limit as $g_i \to \infty$ is not justified. Thus, at present, the conjecture is verified for no family of elliptic curves.

Our main result is the proof of the following fact towards the conjecture.

Theorem 2. 1) *The infinite product for* $L_{\{E_i/K_i\}}(s)$ *converges for* $\operatorname{Re} s \geqslant 1$.

2) *The following formula holds for* $\operatorname{Re} s > 1$*:*

$$\lim_{i\to\infty} \frac{\log L_{E_i}(s)}{g_i} = \log L_{\{E_i/K_i\}}(s).$$

3) *Suppose that the family* E_i/K_i *satisfies the condition* $\lim\limits_{i\to\infty} r_{E_i}/g_i = 0$. *Then*

$$\lim_{i\to\infty} \frac{\log|c_{E_i}|}{g_i} \leqslant \log L_{\{E_i/K_i\}}(1).$$

Remarks. 1) The condition $\lim\limits_{i\to\infty} r_{E_i}/g_i = 0$ holds for every constant family of curves and holds rather often in the general case (see [7]).

2) The problem concerning the equality case in assertion 3) of the theorem is quite subtle and is related to low-placed zeros of the L-functions. For a detailed discussion of the problem (although in a somewhat different situation), see [8].

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Asymptotic properties of the Dedekind zeta function in families of number fields

Our starting point is the classical Brauer—Siegel theorem for number fields proved by Brauer in [1]. It states that if K runs through a sequence of number fields normal over $\mathbb{Q}$ such that $n_K/\log|D_K|\to 0$, then $\log(h_K R_K)/\log\sqrt{|D_K|}\to 1$. Here D_K, h_K, and R_K are respectively the discriminant, the class number, and the regulator of the field K.

In [2] this theorem was generalized by Tsfasman and Vladuts to the case when the condition $n_K/\log|D_K|\to 0$ no longer holds. To formulate this result we will need some notation. For a finite extension $K/\mathbb{Q}$ denote by $\Phi_q(K)$ the number of prime ideals of the ring of integers $\mathcal{O}_K$ having their norm equal to q. Denote by $\Phi_{\mathbb{R}}(K)$ and $\Phi_{\mathbb{C}}(K)$ the number of real and complex embeddings of K into $\mathbb{C}$, respectively. Also let $g_K=\log\sqrt{|D_K|}$ be the genus of the field K (by analogy with the function field case). An extension $K/\mathbb{Q}$ is said to be almost normal if there exists a tower of intermediate extensions $K=K_n\supseteq K_{n-1}\supseteq\cdots\supseteq K_1\supseteq K_0=\mathbb{Q}$ such that K_i/K_{i-1} is normal for all i.

Consider a family $\{K_i\}$ of pairwise non-isomorphic number fields. We define $\phi_\alpha=\lim\limits_{i\to\infty}\Phi_\alpha(K_i)/g_{K_i}$, $\alpha\in\{\mathbb{R},\mathbb{C},2,3,4,5,7,9,\dots\}$. If the limits ϕ_α exist, then the family is said to be asymptotically exact. It is said to be asymptotically good if there exists $\phi_\alpha\neq 0$, and asymptotically bad otherwise.

It is easy to check that the condition $n_K/\log|D_K|\to 0$ in the Brauer—Siegel theorem is equivalent to the condition that the corresponding family be asymptotically bad. We can now formulate the theorem proved by Tsfasman and Vladuts in [2] in the asymptotically good case and by the author in [3] in the asymptotically bad case.

Theorem 1. *For an asymptotically exact family* $\{K_i\}$,

$$\lim_{i\to\infty}\frac{\log(h_{K_i}R_{K_i})}{g_{K_i}}=1+\sum_q\phi_q\log\frac{q}{q-1}-\phi_{\mathbb{R}}\log 2-\phi_{\mathbb{C}}\log(2\pi) \qquad (1)$$

provided that all the K_i *are almost normal over* $\mathbb{Q}$ *or the Generalized Riemann Hypothesis* (*GRH*) *holds for the Dedekind zeta functions of the fields* K_i.

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Let us define the limit zeta function of an asymptotically exact family of number fields as

$$\zeta_{\{K_i\}}(s) = \prod_q (1-q^{-s})^{-\phi_q}.$$

Theorem C in [2] implies the absolute convergence of the infinite product for $\operatorname{Re} s \geqslant 1$. If $\varkappa_K = \operatorname{Res}_{s=1} \zeta_K(s)$ is the residue of the Dedekind zeta function of the field K at the point $s=1$, then the equality (1) can be restated as $\lim_{i\to\infty} \log \varkappa_{K_i}/g_{K_i} = \log \zeta_{\{K_i\}}(1)$. Moreover, it was proved in [2] that $\lim_{i\to\infty} \log \zeta_{K_i}(s)/g_{K_i} = \log \zeta_{\{K_i\}}(s)$ for $\operatorname{Re} s > 1$.

Our main goal is to investigate whether an analogous equality is true for $\operatorname{Re} s < 1$. The case $s=1$ is essentially equivalent to the Brauer—Siegel theorem, and we are at present unable to treat this question in full generality without assuming the GRH. From now on, we assume that the GRH holds for the Dedekind zeta functions of the fields under consideration.

Assuming the GRH, one can prove ([2], the corollary to Theorem A) that the infinite product for $\zeta_{\{K_i\}}(s)$ is absolutely convergent for $\operatorname{Re} s \geqslant 1/2$. We now formulate our main results.

Theorem 2. *Under the assumption of the GRH the equality*

$$\lim_{i\to\infty} \log \zeta_{K_i}(s)/g_{K_i} = \log \zeta_{\{K_i\}}(s)$$

holds for $\operatorname{Re} s > 1/2$ *for an asymptotically exact family* $\{K_i\}$.

The proof of the theorem uses estimates of the logarithmic derivatives of zeta functions in the critical strip together with Vitali's theorem on limits of holomorphic functions.

Our result is weaker for $s=1/2$. We get the following upper estimate:

Theorem 3. *Let* ρ_{K_i} *be the first non-zero coefficient in the Taylor series expansion of* $\zeta_{K_i}(s)$ *at* $s=\frac{1}{2}$, *that is,*

$$\zeta_{K_i}(s) = \rho_{K_i}\Big(s-\frac{1}{2}\Big)^{r_{K_i}} + o\Big(\Big(s-\frac{1}{2}\Big)^{r_{K_i}}\Big).$$

Then under the assumption of the GRH,

$$\lim_{i\to\infty} \log|\rho_{K_i}|/g_{K_i} \leqslant \log \zeta_{\{K_i\}}(1/2)$$

for an asymptotically exact family $\{K_i\}$.

To prove Theorem 3 we employ methods similar to those in the proof of the upper estimate in the equality of Theorem 1 as well as information about the limiting distribution of the zeros of zeta functions on the critical line in families of number fields.

The question of whether equality holds in the statement of Theorem 3 is rather delicate. It is related to the so-called low-lying zeroes of zeta functions, that is, zeroes of $\zeta_K(s)$ having small imaginary parts compared to g_K. We think that the equality $\lim\limits_{i\to\infty} \log|\rho_{K_i}|/g_{K_i} = \log\zeta_{\{K_i\}}(1/2)$ need not be true for all families $\{K_i\}$, since the behaviour of low-lying zeros of zeta functions is rather random. It might, however, be true for 'most' families. A more thorough discussion, though in a slightly different situation (low-lying zeroes of L-functions associated with modular forms on $\mathrm{SL}_2(\mathbb{R})$), can be found in [4].

We formulate a corollary to Theorem 2. Recall that the Euler—Kronecker constant of a number field K is defined as $\gamma_K = c_0(K)/c_{-1}(K)$, where $\zeta_K(s) = c_{-1}(K)(s-1)^{-1} + c_0(K) + O(s-1)$. In [5] Ihara obtained an asymptotic formula for γ in families of curves over finite fields. We have the following result, which is derived from Theorem 2.

Corollary 4. *Under the assumption of the GRH,*

$$\lim_{i\to\infty} \gamma_{K_i}/g_{K_i} = -\sum_q \phi_q \log q/(q-1)$$

for an asymptotically exact family $\{K_i\}$ of number fields.

This was stated in [6] without assuming the GRH. Unfortunately, the proof there is flawed. It uses an unjustified interchange of limits in the sum over prime powers and the limit over the family $\{K_i\}$. Thus, the question of whether such an equality holds without assuming the GRH remains open.

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Asymptotic properties of Dedekind zeta functions in families of number fields

Résumé. Le but de cet article est de démontrer une formule qui exprime la conduite asymptotique de la fonction zêta de Dedekind dans des familles de corps globaux pour $\operatorname{Re} s > 1/2$ en supposant que l'Hypothèse de Riemann Généralisée est vérifiée. On peut voir ce résultat comme une généralization du théorème de Brauer-Siegel. Comme corollaire, on obtient une formule limite pour des constants d'Euler—Kronecker dans des familles de corps globaux.

Abstract. The main goal of this paper is to prove a formula that expresses the limit behaviour of Dedekind zeta functions for $\operatorname{Re} s > 1/2$ in families of number fields, assuming that the Generalized Riemann Hypothesis holds. This result can be viewed as a generalization of the Brauer—Siegel theorem. As an application we obtain a limit formula for Euler—Kronecker constants in families of number fields.

1. Introduction

Our starting point is the classical Brauer—Siegel theorem for number fields first proven by Siegel in the case of quadratic fields and then by Brauer (see [1]) in a more general situation. This theorem states that if K runs through a sequence of number fields normal over $\mathbb{Q}$ such that $n_K / \log |D_K| \to 0$, then $\log(h_K R_K) / \log \sqrt{|D_K|} \to 1$. Here D_K, h_K, R_K and n_K are respectively the discriminant, the class number, the regulator and the degree of the field K.

In [11] this theorem was generalized by Tsfasman and Vlăduţ to the case when the condition $n_K / \log |D_K| \to 0$ no longer holds. To formulate this result we will need to introduce some notation.

For a finite extension $K/\mathbb{Q}$, let $\Phi_q(K)$ be the number of prime ideals of the ring of integers $\mathcal{O}_K$ with norm q, i. e. $\Phi_q(K) = |\{\mathfrak{p} \mid \operatorname{Norm} \mathfrak{p} = q\}|$. Furthermore, denote by $\Phi_{\mathbb{R}}(K)$ and $\Phi_{\mathbb{C}}(K)$ the number of real and complex places of K respectively. Let $g_K = \log \sqrt{|D_K|}$ be the genus of the field K (in analogy with the function field case). An extension $K/\mathbb{Q}$ is called

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almost normal if there exists a tower of extensions $K=K_n\supseteq K_{n-1}\dots$ $\dots\supseteq K_1\supseteq K_0=\mathbb{Q}$ such that K_i/K_{i-1} is normal for all i.

Consider a family of pairwise non-isomorphic number fields $\{K_i\}$.

Definition 1. If the limits

$$\phi_\alpha=\lim_{i\to\infty}\frac{\Phi_\alpha(K_i)}{g_{K_i}},\quad \alpha\in\{\mathbb{R},\mathbb{C},2,3,4,5,7,9,\dots\}$$

exist for each α then the family $\{K_i\}$ is called asymptotically exact. It is asymptotically good if there exists $\phi_\alpha\neq 0$ and asymptotically bad otherwise. The numbers ϕ_α are called the Tsfasman—Vlăduţ invariants of the family $\{K_i\}$.

It is not difficult to check (see [11, Lemma 2.7]) that the condition $n_K/\log|D_K|\to 0$ from the Brauer—Siegel theorem is equivalent to the fact that the corresponding family is asymptotically bad. One can prove that any family contains an asymptotically exact subfamily and that an infinite tower of number fields is always asymptotically exact (see [11, Lemma 2.2 and Lemma 2.4]).

Now we can formulate the Tsfasman—Vlăduţ theorem proven in [11, Theorem 7.3] in the asymptotically good case and in [12, Theorem 1] in the asymptotically bad one.

Theorem 1. *For an asymptotically exact family $\{K_i\}$ we have*

$$\lim_{i\to\infty}\frac{\log(h_{K_i}R_{K_i})}{g_{K_i}}=1+\sum_q\phi_q\log\frac{q}{q-1}-\phi_\mathbb{R}\log 2-\phi_\mathbb{C}\log 2\pi,\tag{1.1}$$

provided either all K_i are almost normal over $\mathbb{Q}$ or the Generalized Riemann Hypothesis (*GRH*) *holds for zeta functions of the fields K_i.*

To generalize this theorem still further we will have to use the concept of limit zeta functions from [11].

Definition 2. The limit zeta function of an asymptotically exact family of number fields $\{K_i\}$ is defined as

$$\zeta_{\{K_i\}}(s)=\prod_q(1-q^{-s})^{-\phi_q}.$$

Theorem C from [11] gives us the convergence of the above infinite product for $\operatorname{Re}s\geqslant 1$. Let $\varkappa_K=\operatorname{Res}_{s=1}\zeta_K(s)$ be the residue of the Dedekind zeta function of the field K at $s=1$. Using the residue formula (see [7, Chapter VIII, Theorem 5])

$$\varkappa_K=\frac{2^{\Phi_\mathbb{R}(K)}(2\pi)^{\Phi_\mathbb{C}(K)}h_KR_K}{w_K\sqrt{|D_K|}}$$

(here w_K is the number of roots of unity in K) and the estimate $w_K = O(n_K^2)$ (see [7, p. 322]) one can see that the question about the behaviour of the ratio from the Brauer—Siegel theorem is immediately reduced to the corresponding question for $\varkappa_K$.

The formula (1.1) can be rewritten as $\lim\limits_{i\to\infty} \dfrac{\log \varkappa_{K_i}}{g_{K_i}} = \log \zeta_{\{K_i\}}(1)$. Furthermore, Tsfasman and Vlăduţ prove in [11, Proposition 4.2] that for $\operatorname{Re} s > 1$ the equality $\lim\limits_{i\to\infty} \dfrac{\log \zeta_{K_i}(s)}{g_{K_i}} = \log \zeta_{\{K_i\}}(s)$ holds.

Our main goal is to investigate the question of the validity of the above equality for $\operatorname{Re} s < 1$. We work in the number field case, for the function field case see [13], where the same problem was treated in a much broader context.

The case $s = 1$ is in a sense equivalent to the Brauer—Siegel theorem so current techniques does not allow to treat it in full generality without the assumption of GRH. From now on we will assume that GRH holds for Dedekind zeta functions of the fields under consideration. Assuming GRH, Tsfasman and Vlăduţ proved ([11, Corollary from Theorem A]) that the infinite product for $\zeta_{\{K_i\}}(s)$ is absolutely convergent for $\operatorname{Re} s \geqslant \frac{1}{2}$. We can now formulate our main results.

Theorem 2. *Assuming GRH, for an asymptotically exact family of number fields $\{K_i\}$ for $\operatorname{Re} s > \frac{1}{2}$ we have*

$$\lim_{i\to\infty} \frac{\log((s-1)\zeta_{K_i}(s))}{g_{K_i}} = \log \zeta_{\{K_i\}}(s).$$

The convergence is uniform on compact subsets of the half-plane $\left\{s \mid \operatorname{Re} s > \frac{1}{2}\right\}$.

The result for $s = \frac{1}{2}$ is considerably weaker and we can only prove the following upper bound:

Theorem 3. *Let ρ_{K_i} be the first non-zero coefficient in the Taylor series expansion of $\zeta_{K_i}(s)$ at $s = \frac{1}{2}$, i. e.*

$$\zeta_{K_i}(s) = \rho_{K_i}\Big(s - \frac{1}{2}\Big)^{r_{K_i}} + o\Big(\Big(s - \frac{1}{2}\Big)^{r_{K_i}}\Big).$$

Then, assuming GRH, for any asymptotically exact family of number fields $\{K_i\}$ the following inequality holds:

$$\limsup_{i\to\infty} \frac{\log |\rho_{K_i}|}{g_{K_i}} \leqslant \log \zeta_{\{K_i\}}\Big(\frac{1}{2}\Big). \tag{1.2}$$

The question whether the equality holds in Theorem 3 is rather delicate. It is related to the so called low-lying zeroes of zeta functions, that is the zeroes of $\zeta_K(s)$ having small imaginary part compared to g_K. We doubt that the equality $\lim\limits_{i\to\infty}\dfrac{\log|\rho_{K_i}|}{g_{K_i}}=\log\zeta_{\{K_i\}}\left(\dfrac{1}{2}\right)$ holds for any asymptotically exact family $\{K_i\}$ since the behaviour of low-lying zeroes is known to be rather random. Nevertheless, it might hold for "most" families (whatever it might mean). A more thorough discussion of this question in a slightly different situation (low-lying zeroes of L-functions of modular forms on $\mathrm{SL}_2(\mathbb{R})$) can be found in [4].

To illustrate how hard the problem may be, let us remark that Iwaniec and Sarnak studied a similar question for the central values of L-functions of Dirichlet characters [5] and modular forms [6]. They manage to prove that there exists a positive proportion of Dirichlet characters (modular forms) for which the logarithms of the central values of the corresponding L-functions divided by the logarithms of the analytic conductors tend to zero. The techniques of the evaluation of mollified moments used in these papers are rather involved. We also note that, to our knowledge, there has been no investigation of low-lying zeroes of L-functions of growing degree. It seems that the analogous problem in the function field has neither been very well studied.

Let us formulate a corollary of the Theorem 2. We will need the following definition:

Definition 4. The Euler—Kronecker constant of a number field K is defined as $\gamma_K=\dfrac{c_0(K)}{c_{-1}(K)}$, where $\zeta_K(s)=c_{-1}(K)(s-1)^{-1}+c_0(K)+O(s-1)$.

Ihara made an extensive study of the Euler—Kronecker constant in [2]. In particular, he obtained an asymptotic formula for the behaviour of γ in families of curves over finite fields. As a corollary of Theorem 2, we prove the following analogue of Ihara's result in the number field case:

Corollary 1. *Assuming GRH, for any asymptotically exact family of number fields $\{K_i\}$ we have*

$$\lim_{i\to\infty}\frac{\gamma_{K_i}}{g_{K_i}}=-\sum_q\phi_q\frac{\log q}{q-1}.$$

This result was formulated in [10] without the assumption of the Riemann hypothesis. Unfortunately, the proof given there is flawed. It uses an unjustified change of limits in the summation over prime powers and the limit taken over the family $\{K_i\}$. Thus, the question about the validity of this equality without the assumption of GRH is still open.

It would be interesting to have a result of this type at least under a certain normality condition on our family $\{K_i\}$. Even the study of abelian extensions is not uninteresting in this setting.

2. Proofs of the main results

Proof of Theorem 2. The statement of the theorem is known for $\operatorname{Re} s > 1$ (see [11, Proposition 4.2]) thus we can freely assume that $\operatorname{Re} s < 2$.

We will use the following well known result [3, Proposition 5.7] which can be proven using Hadamard's factorization theorem.

Proposition 1. (1) *For* $-\frac{1}{2} \leqslant \sigma \leqslant 2$, $s = \sigma + it$ *we have*

$$\frac{\zeta_K'(s)}{\zeta_K(s)} + \frac{1}{s} + \frac{1}{s-1} - \sum_{|s-\rho|<1} \frac{1}{s-\rho} = O(g_K),$$

where ρ runs through all non-trivial zeroes of $\zeta_K(s)$ and the constant in O is absolute.

(2) *The number $m(T,K)$ of zeroes $\rho = \beta + \gamma i$ of $\zeta_K(s)$ such that $|\gamma - T| \leqslant 1$ satisfies $m(T,K) < C(g_K + n_K \log(|T|+4))$ with an absolute constant C.*

Now, applying this proposition, we see that for fixed $T > 0$, $\varepsilon > 0$ and any $s \in \mathcal{D}_{T,\varepsilon} = \{s \in \mathbb{C} \mid |\operatorname{Im} s| \leqslant T,\ \varepsilon + \frac{1}{2} \leqslant \operatorname{Re} s \leqslant 2\}$ we have

$$\frac{\zeta_K'(s)}{\zeta_K(s)} + \frac{1}{s-1} = \sum_{|s-\rho|<\varepsilon} \frac{1}{s-\rho} + O_{T,\varepsilon}(g_K), \tag{2.1}$$

for by Minkowski's theorem [7, Chapter V, Theorem 4] $n_K < C g_K$ with an absolute constant C.

If we assume GRH, the sum over zeroes on the right hand side of (2.1) disappears. Integrating, we finally get that in $\mathcal{D}_{T,\varepsilon}$

$$\frac{\log(\zeta_K(s)(s-1))}{g_K} = O_{T,\varepsilon}(1)$$

Now, we can use the so called Vitali theorem [9, 5.21]:

Proposition 2. *Let $f_n(s)$ be a sequence of functions holomorphic in a domain $\mathcal{D}$. Assume that for some $M \in \mathbb{R}$ we have $|f_n(s)| < M$ for any n and $s \in \mathcal{D}$. Let also $f_n(s)$ tend to a limit at a set of points having a limit point in $\mathcal{D}$. Then the sequence $f_n(s)$ tends to a holomorphic function in $\mathcal{D}$ uniformly on any closed disk contained in $\mathcal{D}$.*

It suffices to notice that the convergence of $\log \zeta_{K_i}(s)/g_{K_i}$ to $\zeta_{\{K_i\}}(s)$ is known for $\operatorname{Re} s > 1$ by [11, Proposition 4.2]. So, applying the above theorem and using the fact that under GRH $\zeta_{\{K_i\}}(s)$ is holomorphic for $\operatorname{Re} s \geqslant \frac{1}{2}$ [11, corollary from Theorem A] we get the required result. $\square$

Proof of Theorem 3. Denote $g_k = g_{K_k}$. Let us write

$$\zeta_{K_k}(s) = c_k\Big(s-\frac{1}{2}\Big)^{r_k} F_k(s),$$

where $F_k(s)$ is an analytic function in the neighbourhood of $s=\frac{1}{2}$ such that $F_k\Big(\frac{1}{2}\Big)=1$. Let us put $s=\frac{1}{2}+\theta$, where $\theta>0$ is a small positive real number. We have

$$\frac{\log \zeta_{K_k}(\frac{1}{2}+\theta)}{g_k} = \frac{\log c_k}{g_k} + r_k\frac{\log\theta}{g_k} + \frac{\log F_k(\frac{1}{2}+\theta)}{g_k}.$$

To prove the theorem we will construct a sequence θ_k such that

(1) $\frac{1}{g_k}\log\zeta_{K_k}\Big(\frac{1}{2}+\theta_k\Big)\to\log\zeta_{\{K_k\}}\Big(\frac{1}{2}\Big)$;

(2) $\frac{r_k}{g_k}\log\theta_k\to 0$;

(3) $\liminf \frac{1}{g_k}\log F_k\Big(\frac{1}{2}+\theta_k\Big)\geqslant 0$.

For each natural number N we choose $\theta(N)$ a decreasing sequence such that

$$\Big|\zeta_{\{K_k\}}\Big(\frac{1}{2}\Big)-\zeta_{\{K_k\}}\Big(\frac{1}{2}+\theta(N)\Big)\Big| < \frac{1}{2N}.$$

This is possible since $\zeta_{\{K_k\}}(s)$ is continuous for $\operatorname{Re} s\geqslant\frac{1}{2}$ by [11, corollary from Theorem A]. Next, we choose a sequence $k'(N)$ with the property:

$$\Big|\frac{1}{g_k}\log\zeta_{K_k}\Big(\frac{1}{2}+\theta\Big)-\log\zeta_{\{K_k\}}\Big(\frac{1}{2}+\theta\Big)\Big| < \frac{1}{2N}$$

for any $\theta\in[\theta(N+1),\theta(N)]$ and any $k\geqslant k'(N)$. This is possible by Theorem 2. Then we choose $k''(N)$ such that

$$\frac{-r_k\log\theta(N+1)}{g_k}\leqslant\frac{\theta(N)}{N}$$

for any $k\geqslant k''(N)$, which can be done thanks to the following proposition (c.f. [3, Proposition 5.34]):

Proposition 3. *Assume that GRH holds for $\zeta_K(s)$. Then*

$$\operatorname*{ord}_{s=\frac{1}{2}}\zeta_K(s) < \frac{C\log 3|D_K|}{\log\log 3|D_K|}$$

the constant C being absolute.

Finally, we choose an increasing sequence $k(N)$ such that $k(N) \geqslant \max(k'(N), k''(N))$ for any N.

Now, if we define $N = N(k)$ by the inequality $k(N) \leqslant k \leqslant k(N+1)$ and let $\theta_k = \theta(N(k))$, then from the conditions imposed on θ_k we automatically get (1) and (2). The delicate point is (3). We will use Hadamard's product formula [8, p. 137]:

$$\log|D_K| = \Phi_{\mathbb{R}}(K)(\log\pi - \psi(s/2)) + \\ + 2\Phi_{\mathbb{C}}(K)(\log(2\pi) - \psi(s)) - \frac{2}{s} - \frac{2}{s-1} + 2\sum_{\rho}{}' \frac{1}{s-\rho} - 2\frac{\zeta'_K(s)}{\zeta_K(s)},$$

where $\psi(s) = \Gamma'(s)/\Gamma(s)$ is the logarithmic derivative of the gamma function. In the first sum ρ runs over the zeroes of $\zeta_K(s)$ in the critical strip and $\sum'$ means that ρ and $\bar\rho$ are to be grouped together. This can be rewritten as

$$\frac{1}{g_k}\Bigl(\log\zeta_k\Bigl(\frac{1}{2}+\theta\Bigr) - r_k\log\theta\Bigr)' = -1 + \frac{\Phi_{\mathbb{R}}(K_k)}{2g_k}\Bigl(\log\pi - \psi\Bigl(\frac{1}{4}+\frac{\theta}{2}\Bigr)\Bigr) + \\ + \frac{\Phi_{\mathbb{C}}(K_k)}{g_k}\Bigl(\log 2\pi - \psi\Bigl(\frac{1}{2}+\theta\Bigr)\Bigr) + \frac{8\theta}{(1-4\theta^2)g_k} + \sum_{\rho\neq 1/2}{}' \frac{1}{(1/2+\theta-\rho)g_k}.$$

(the term $r_k\log\theta$ comes from the contribution of zeroes at $s = \frac{1}{2}$). One notices that all the terms on the right hand side except for -1 and $\frac{8\theta}{(1-4\theta^2)g_k}$ are positive. Thus, we see that $\frac{1}{g_k}\Bigl(\log F_k\Bigl(\frac{1}{2}+\theta\Bigr)\Bigr)' \geqslant C$ for any small enough θ, where C is an absolute constant. From this and from the fact that $F_k\Bigl(\frac{1}{2}\Bigr) = 1$ we deduce that

$$\frac{1}{g_k}\log F_k\Bigl(\frac{1}{2}+\theta_k\Bigr) \geqslant C\theta_k \to 0.$$

This proves (3) as well as the theorem. $\square$

Proof of the Corollary 1. It suffices to take the values at $s = 1$ of the derivatives of both sides of the equality in Theorem 2. This is possible since the convergence is uniform for $\operatorname{Re} s > \frac{1}{2}$. $\square$

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Jacobians among abelian threefolds: a formula of Klein and a question of Serre

(with G. Lachaud and C. Ritzenthaler)

Abstract. In this paper we give a criterion when an indecomposable principally polarized abelian threefold (A, a) defined over a field $k \subset \mathbb{C}$ is a Jacobian over k. More precisely, (A, a) is a Jacobian over k if and only if the value of a certain geometric Siegel modular form $\chi_{18}(A, a)$ is a square over k. This answers a question of J.-P. Serre.

1. Introduction

Let k be an algebraically closed field and let $g \geqslant 1$ be an integer. If X is a (nonsingular, irreducible, projective) curve of genus g over k, Torelli's theorem states that the map $X \mapsto (\operatorname{Jac} X, j)$, associating to X its Jacobian together with the canonical polarization j, is injective. The determination of the image of this map is a long time studied question.

When $g = 3$, the moduli space A_g of principally polarized abelian varieties of dimension g and the moduli space M_g of nonsingular algebraic curves of genus g are both of dimension $g(g+1)/2 = 3g - 3 = 6$. According to Hoyt [4] and Oort and Ueno [10], the image of M_3 is exactly the space of indecomposable principally polarized abelian threefolds. Moreover, if $k = \mathbb{C}$ Igusa [5] characterized the locus of decomposable abelian threefolds and that of hyperelliptic Jacobians, making use of two particular modular forms Σ_{140} and χ_{18} on the Siegel upper half space of degree 3. A similar characterization also exists in case $g = 2$ (c. f. [9]).

Assume now that k is any field and $g \geqslant 1$. J.-P. Serre noticed in [8] that a precise form of Torelli's theorem reveals a mysterious obstruction for a geometric Jacobian to be a Jacobian over k. More precisely, he proved the following:

Theorem 1. *Let (A, a) be a principally polarized abelian variety of dimension $g \geqslant 1$ over k, and assume that (A, a) is isomorphic over $\bar{k}$ to the*

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Jacobian of a curve X_0 of genus g defined over $\bar{k}$. The following alternative holds:

1) *If X_0 is hyperelliptic, there is a curve X/k isomorphic to X_0 over $\bar{k}$ such that (A, a) is k-isomorphic to* $(\operatorname{Jac} X, j)$.

2) *If X_0 is not hyperelliptic, there is a curve X/k isomorphic to X_0 over $\bar{k}$, and a quadratic character*

$$\varepsilon\colon \operatorname{Gal}(k^{\text{sep}}/k) \to \{\pm 1\}$$

such that the twisted abelian variety $(A, a)_\varepsilon$ is k-isomorphic to $(\operatorname{Jac} X, j)$. *The character ε is trivial if and only if (A, a) is k-isomorphic to a Jacobian.*

Thus, only case 1) occurs if $g = 1$ or $g = 2$, with all curves being elliptic or hyperelliptic. In this article we completely resolve for fields of characteristic zero the first previously unknown case $g = 3$.

Let there be given an indecomposable principally polarized abelian threefold (A, a) defined over k. In a letter to J. Top [11], J.-P. Serre asked a twofold question:

- *How to decide, knowing only (A, a), that X is hyperelliptic?*
- *If X is not hyperelliptic, how to compute the quadratic character ε?*

Assume that $k \subset \mathbb{C}$. The first question can easily be answered using the modular forms Σ_{140} and χ_{18}. As for the second question, roughly speaking, Serre suggested that ε is trivial if and only if χ_{18} is a square in $k^\times$ (see Theorem 2 for a more precise formulation). This assertion was motivated by a formula of Klein [6] relating the modular form χ_{18} (in the notation of Igusa) to squares of discriminants of plane quartics, which more or less gives the 'only if' part of the claim. In [7], two of the authors justified Serre's assertion for a particular three dimensional family of abelian varieties and in particular determined the absolute constant involved in Klein's formula.

In this article we justify Serre's assertion for any abelian threefold, thus giving an algorithm which allows to determine whether a given principally polarized abelian threefold over k is a Jacobian over k. In order to do so, we start by taking a broader point of view, valid for any $g > 1$.

1) We look at the action of $\bar{k}$-isomorphisms on Siegel modular forms defined over k and we define invariants of k-isomorphism classes of abelian varieties over k.

2) We link Siegel modular forms, Teichmüller modular forms and invariants of plane curves.

Once these two goals are achieved, Serre's assertion can be rephrased as the following strategy:

- use 2) to prove that a certain Siegel modular form f is a suitable n-th power with $n>1$ on the Jacobian locus;
- use 1) to distinguish between Jacobians and their twists. Indeed, the action of a twist on f may change its value by a non n-th power and then, according to 2) of Theorem 1, we have a criterion to distinguish Jacobians.

For $g=3$, Klein's formula shows that the form χ_{18} is a square on the Jacobian locus and that this is enough to characterize this locus. The relevance of Klein's formula in this problem was one of Serre's insights. We would like to point out that we do not actually need the full strength of Klein's formula to work out our strategy. One can use instead a formula due to Ichikawa relating χ_{18} to the square of a Teichmüller modular form, denoted $\mu_{3,9}$. However we think that the connection between Siegel modular forms and invariants is interesting enough on its own, besides the fact that it gives a new rigorous proof of Klein's formula.

2. Main theorems

Let $\mathbb{H}_g = \left\{\tau \in \mathbf{M}_g(\mathbb{C}) \mid {}^t\tau = \tau,\ \operatorname{Im}\tau > 0\right\}$ be the Siegel upper half space of genus g.

We recall the definition of theta functions with (entire) characteristics

$$[\boldsymbol{\varepsilon}] = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \in \mathbb{Z}^g \oplus \mathbb{Z}^g,$$

following [1]. The (*classical*) *theta function* is given, for $\tau \in \mathbb{H}_g$ and $z \in \mathbb{C}^g$, by

$$\theta\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}(z,\tau) = \sum_{n\in\mathbb{Z}^g} e^{2\pi i((n+\varepsilon_1/2)\tau(n+\varepsilon_1/2)+2(n+\varepsilon_1/2)(z+\varepsilon_2/2))}.$$

The *Thetanullwerte* are the values at $z=0$ of these functions, and we write

$$\theta[\boldsymbol{\varepsilon}](\tau) = \theta\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}(\tau) = \theta\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}(0,\tau).$$

Recall that a characteristic is *even* if $\varepsilon_1 \cdot \varepsilon_2 \equiv 0 \pmod 2$ and *odd* otherwise. Let S_g be the set of even characteristics with coefficients in $\{0,1\}$. For $g \geqslant 2$, we put $h = |S_g|/2 = 2^{g-2}(2^g+1)$ and

$$\widetilde{\chi}_h(\tau) = \prod_{\boldsymbol{\varepsilon}\in S_g} \theta[\boldsymbol{\varepsilon}](\tau).$$

Denote by $\widetilde{\Sigma}_{140}$ the modular form defined by the thirty-fifth elementary symmetric function of the eighth powers of the even Thetanullwerte.

Recall that a principally polarized abelian variety (A, a) is decomposable if it is a product of principally polarized abelian varieties of lower dimensions, and it is indecomposable otherwise.

Let $k \subset \mathbb{C}$ be a field and let $g=3$. Consider a principally polarized abelian threefold (A, a) defined over k. Let $\omega_1, \omega_2, \omega_3$ be any basis of the space of differential forms $\Omega^1_k[A] = H^0(A, \Omega^1_A)$ and let $\gamma_1, \ldots, \gamma_6$ be a symplectic basis (for the polarization a) of $H_1(A, \mathbb{Z})$, in such a way that

$$\Omega = [\Omega_1\ \Omega_2] = \begin{pmatrix} \int_{\gamma_1} \omega_1 & \cdots & \int_{\gamma_6} \omega_1 \\ \vdots & & \vdots \\ \int_{\gamma_1} \omega_3 & \cdots & \int_{\gamma_6} \omega_3 \end{pmatrix}$$

is a period matrix of (A, a). Put $\tau = \Omega_2^{-1}\Omega_1 \in \mathbb{H}_3$.

We have the following theorem which allows us to determine whether a given abelian threefold defined over k is k-isomorphic to a Jacobian of a curve defined over the same field. This settles the question of Serre recalled in the introduction.

Theorem 2. 1) *If* $\widetilde{\Sigma}_{140}(\tau) = 0$ *and* $\widetilde{\chi}_{18}(\tau) = 0$ *then* (A, a) *is decomposable over* $\bar{k}$. *In particular it is not a Jacobian.*

2) *If* $\widetilde{\Sigma}_{140}(\tau) \neq 0$ *and* $\widetilde{\chi}_{18}(\tau) = 0$ *then there exists a hyperelliptic curve* X/k *such that* $(\operatorname{Jac} X, j) \simeq (A, a)$.

3) *If* $\widetilde{\chi}_{18}(\tau) \neq 0$ *then* (A, a) *is isomorphic to a Jacobian if and only if*

$$-\chi_{18}(A, \omega_1 \wedge \omega_2 \wedge \omega_3) = (2\pi)^{54} \frac{\widetilde{\chi}_{18}(\tau)}{\det(\Omega_2)^{18}}$$

is a square in k.

Corollary 3. *In the notation of Theorem 2, the quadratic character* ε *of* $\operatorname{Gal}(k^{\mathrm{sep}}/k)$ *introduced in Theorem 1 is given by* $\varepsilon(\sigma) = d/d^{\sigma}$, *where*

$$d = \sqrt{(2\pi)^{54} \frac{\widetilde{\chi}_{18}(\tau)}{\det(\Omega_2)^{18}}},$$

with an arbitrary choice of the square root.

Our proof of the theorem is based on the so called Klein's formula which has an interest in itself. To formulate this result we have to introduce yet another notation.

Let F be a homogeneous polynomial of degree 4 in 3 variables x_1, x_2, x_3 with coefficients from the field k and let C_F be the corresponding quartic in $\mathbb{P}^2$. It is well known [3, Chapter 9, Example 1.6(a)] that up to a sign there exists a unique polynomial $\operatorname{Disc} F$ in coefficients of F, irreducible over $\mathbb{Z}$, such that $\operatorname{Disc} F = 0$ if and only if C_F is singular.

Assume that C_F is non-singular. We recall the classical way to write down an explicit k-basis of $\Omega^1[C_F] = H^0(C_F, \Omega^1)$ (see [2, p. 630]). Let

$$\eta_1 = \frac{f(x_2dx_3 - x_3dx_2)}{\partial F/\partial x_1}, \quad \eta_2 = \frac{f(x_3dx_1 - x_1dx_3)}{\partial F/\partial x_2}, \quad \eta_3 = \frac{f(x_1dx_2 - x_2dx_1)}{\partial F/\partial x_3},$$

where f is a linear polynomial in x_1, x_2, x_3. The forms η_i glue together and define a regular differential form $\eta_f(F) \in \Omega^1[C_F]$. Now, denote by η_1, η_2, η_3 the sequence of sections obtained by substituting x_1, x_2, x_3 for f in η_f.

Let $\gamma_1, \ldots, \gamma_6$ be a symplectic basis of $H_1(C_F, \mathbb{Z})$ for the intersection pairing. Let

$$\Omega = [\Omega_1\ \Omega_2] = \begin{pmatrix} \int\limits_{\gamma_1} \eta_1 & \cdots & \int\limits_{\gamma_6} \eta_1 \\ \vdots & & \vdots \\ \int\limits_{\gamma_1} \eta_3 & \cdots & \int\limits_{\gamma_6} \eta_3 \end{pmatrix}$$

be a period matrix of $\operatorname{Jac}(C)$ and let $\tau = \Omega_2^{-1}\Omega_1 \in \mathbb{H}_3$.

Our second main result is the following one:

Theorem 4 (Klein's formula)**.** *In the above notation we have*

$$\operatorname{Disc}(F)^2 = \frac{1}{2^{28}}(2\pi)^{54}\frac{\widetilde{\chi}_{18}(\tau)}{\det(\Omega_2)^{18}}.$$

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Jacobians among Abelian threefolds: a formula of Klein and a question of Serre

(with G. Lachaud and C. Ritzenthaler)

Abstract. Let (A,a) be an indecomposable principally polarized abelian threefold defined over a field $k \subset \mathbb{C}$. Using a certain geometric Siegel modular form χ_{18} on the corresponding moduli space, we prove that (A,a) is a Jacobian over k if and only if $\chi_{18}(A,a)$ is a square over k. This answers a question of J.-P. Serre. Then, via a natural isomorphism between invariants of ternary quartics and Teichmüller modular forms of genus 3, we obtain a simple proof of Klein formula, which asserts that $\chi_{18}(\operatorname{Jac} C, j)$ is equal to the square of the discriminant of C.

Introduction

Let A_3 be the moduli stack of principally polarized abelian schemes (A,a) of relative dimension 3 and M_3 be the moduli stack of smooth and proper curves of genus 3. The first aim of this article is to answer the following question of Serre [20]: If k is a subfield of $\mathbb{C}$, and if $(A,a) \in \mathsf{A}_3 \otimes k$, under what conditions is it isomorphic over k to a polarized Jacobian? If $k = \bar{k}$, this is the case if and only if (A,a) is indecomposable, according to Hoyt [9] and Oort and Ueno [18]. We henceforth assume (A,a) indecomposable, and isomorphic over $\bar{k}$ to the principally polarized Jacobian $(\operatorname{Jac} C, j)$ of a curve $C/\bar{k}$ of genus 3. For a general field $k \subset \mathbb{C}$, the answer is given by a particular Siegel modular form χ_{18} of genus 3. This form is actually defined up to a multiplicative constant by the product of the 36 Thetanullwerte with even characteristics. Our main result (Th. 1.3.3) is the following criterion: (A,a) is isomorphic over k to $(\operatorname{Jac} C, j)$ if and only if $\chi_{18}(A,a)$ is a square over k. This was suggested by Serre in [20] and proved in [15] by the first two authors for a 3-dimensional family of abelian varieties. This square appears due to the following fact: by taking the inverse image under the

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Torelli morphism $t\colon C \mapsto (\operatorname{Jac} C, j)$ (with j the canonical polarization), we get an element $t^*\chi_{18}$ in the algebra of Teichmüller modular forms over k, which turns out to be a square, according to Ichikawa [10]. The equivalence is then obtained by the action of quadratic twists on geometric Siegel modular forms.

The second part of the article uses a natural isomorphism between the algebra of invariants on the space of ternary quartic forms with non zero discriminant and the algebra of Teichmüller modular forms on the space of non hyperelliptic curves of genus 3. Hence, the form $t^*\chi_{18}$, restricted to nonsingular non hyperelliptic curves, can be interpreted as an invariant and this provides a simple proof of a formula of Klein, which asserts that $\chi_{18}(\operatorname{Jac} C, j)$ is the square of the discriminant of C (Th. 2.2.3). The original relevance of Klein's formula for the above criterion was one of Serre's insights.

This article is organized in two sections. In § 1.1, we review the necessary elements from the theory of Siegel and Teichmüller modular forms, then in § 1.2 we introduce the action of isomorphisms and see how the action of twists is reflected on the values of modular forms, and we prove our main result in § 1.3. The second section deals with invariants: in § 2.1, we give a geometric description of invariants of ternary forms, and in § 2.2, we prove Klein's formula. Finally, in § 2.3 we discuss the reasons behind the failure of a straightforward generalization of the theory in higher dimensions.

We would like to thank J.-P. Serre and S. Meagher for fruitful discussions, and Y. F. Bilu and X. Xarles for their help in § 2.3.

1. Modular forms and abelian threefolds

1.1. Siegel and Teichmüller modular forms

References for the following results are [3], [4], [5], [7]. Let $g \geqslant 2$ be an integer and A_g be the moduli stack of principally polarized abelian schemes of relative dimension g. Let $\pi\colon \mathsf{V}_g \to \mathsf{A}_g$ be the universal abelian scheme and $\pi_*\Omega^1_{\mathsf{V}_g/\mathsf{A}_g} \to \mathsf{A}_g$ the rank g bundle, usually called *Hodge bundle*, induced by the relative regular differential forms of degree one on V_g over A_g. The relative canonical bundle over A_g is the line bundle

$$\boldsymbol{\omega} = \bigwedge^g \pi_*\Omega^1_{\mathsf{V}_g/\mathsf{A}_g}.$$

Let R be a commutative ring and h be a positive integer. A *geometric Siegel modular form* of genus g and weight h over R is an element of the R-module

$$\mathbf{S}_{g,h}(R) = \Gamma(\mathsf{A}_g \otimes R, \boldsymbol{\omega}^{\otimes h}).$$

One proceeds similarly with curves. Let M_g denote the moduli stack of smooth and proper curves of genus g. Let $\pi\colon \mathsf{C}_g \to \mathsf{M}_g$ be the universal curve, and let $\boldsymbol{\lambda}$ be the invertible sheaf associated to the *Hodge bundle* on M_g, namely

$$\boldsymbol{\lambda} = \bigwedge^{g} \pi_* \Omega^1_{\mathsf{C}_g/\mathsf{M}_g}.$$

A *Teichmüller modular form* of genus g and weight h over R is an element of

$$\mathbf{T}_{g,h}(R) = \Gamma(\mathsf{M}_g \otimes R, \boldsymbol{\lambda}^{\otimes h}).$$

Assume now that $R = k$ is a field. For a projective nonsingular variety X defined over k, we denote by $\Omega^1_k[X] = H^0(X, \Omega^1_X \otimes k)$ the finite dimensional k-vector space of regular differential forms on X defined over k. Let $(A, a) \in \mathsf{A}_g \otimes k$ be a principally polarized abelian variety of dimension g defined over k (resp. $C \in \mathsf{M}_g \otimes k$ a genus g curve defined over k). We denote by

$$\omega_k[A] \simeq \bigwedge^{g} \Omega^1_k[A] \quad (\text{resp. } \lambda_k[C] \simeq \bigwedge^{g} \Omega^1_k[C])$$

the k-vector space of sections of $\boldsymbol{\omega}$ (resp. $\boldsymbol{\lambda}$) over (A, a) (resp. C). For $f \in \mathbf{S}_{g,h}(k)$ (resp. $f \in \mathbf{T}_{g,h}(k)$), and ω a basis of $\omega_k[A]$ (resp. λ a basis of $\lambda_k[C]$), we put

$$f((A, a), \omega) = f(A, a)/\omega^{\otimes h} \in k, \quad (\text{resp. } f(C, \lambda) = f(C)/\lambda^{\otimes h} \in k). \tag{1}$$

In this way a modular form defines a rule which assigns the element $f((A, a), \omega) \in k$ (resp. $f(C, \lambda)$) to every such pair $((A, a), \omega)$ (resp. (C, λ)) which depends only on $\bar{k}$-isomorphism class of the pair. With this definition, the following proposition holds, see for instance [11]:

Proposition 1.1.1. *The Torelli map $t\colon \mathsf{M}_g \to \mathsf{A}_g$, associating to a curve C its Jacobian $\operatorname{Jac} C$ with the canonical polarization j, satisfies $t^*\boldsymbol{\omega} = \boldsymbol{\lambda}$, and induces for any field k a linear map*

$$t^*\colon \mathbf{S}_{g,h}(k) = \Gamma(\mathsf{A}_g \otimes k, \boldsymbol{\omega}^{\otimes h}) \longrightarrow \mathbf{T}_{g,h}(k) = \Gamma(\mathsf{M}_g \otimes k, \boldsymbol{\lambda}^{\otimes h}),$$

For any curve C/k of genus g and any $f \in \mathbf{S}_{g,h}(k)$, one has $[t^*f](C) = = t^*[f(\operatorname{Jac} C, j)]$, i. e. for any basis ω of $\omega_k[\operatorname{Jac} C]$,

$$f((\operatorname{Jac} C, j), \omega) = [t^*f](C, \lambda) \quad \text{if } t^*\omega = \lambda. \qquad \square$$

Assume now that $R=k=\mathbb{C}$. Let $\mathbf{R}_{g,h}(\mathbb{C})$ be the vector-space of *analytic Siegel modular forms* of weight h on $\mathbb{H}_g=\{\tau\in\mathbf{M}_g(\mathbb{C})^t \,\big|\, \tau=\tau, \operatorname{Im}\tau>0\}$, consisting of complex holomorphic functions $\phi(\tau)$ on $\mathbb{H}_g$ satisfying

$$\phi(M.\tau)=\det(c\tau+d)^h\cdot\phi(\tau)\quad \text{if } M=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \operatorname{Sp}_{2g}(\mathbb{Z}).$$

To a point $\tau\in\mathbb{H}_g$ we associate the abelian variety $A_\tau=\mathbb{C}^g/(\mathbb{Z}^g+\tau\mathbb{Z}^g)$ with its natural principal polarization j. Since the tangent space to A_τ is canonically isomorphic to $\mathbb{C}^g$, $dz_1\wedge\cdots\wedge dz_g$ is a section of

$$\boldsymbol{\omega}\otimes\mathbb{C}\simeq\mathscr{O}_{\mathbb{H}_g}\otimes\bigwedge^g(\mathbb{C}^g).$$

Thus, it induces a map from $\mathbf{R}_{g,h}(\mathbb{C})$ to $\mathbf{S}_{g,h}(\mathbb{C})$. More precisely, the following result holds [5, p. 141]:

Proposition 1.1.2. *If $f\in\mathbf{S}_{g,h}(\mathbb{C})$ and $\tau\in\mathbb{H}_g$, let*

$$\widetilde{f}(\tau)=(2i\pi)^{-gh}f(A_\tau,j)/(dz_1\wedge\cdots\wedge dz_g)^{\otimes h}$$

where $(z_1,\dots z_g)$ is the canonical basis of $\mathbb{C}^g$. The map $f\mapsto\widetilde{f}$ is an isomorphism $\mathbf{S}_{g,h}(\mathbb{C})\xrightarrow{\sim}\mathbf{R}_{g,h}(\mathbb{C})$. □

In the sequel we shall need some specific Siegel modular forms. We recall the definition of Thetanullwerte with characteristics

$$\boldsymbol{\varepsilon}=\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\end{bmatrix}\in\{0,1\}^g\oplus\{0,1\}^g,$$

given, for $\tau\in\mathbb{H}_g$, by

$$\theta[\boldsymbol{\varepsilon}](\tau)=\sum_{n\in\mathbb{Z}^g}\exp\Big(i\pi\Big(n+\frac{\varepsilon_1}{2}\Big)\tau^t(n+\varepsilon_1/2)+2i\pi\Big(n+\frac{\varepsilon_1}{2}\Big)^t(\varepsilon_2/2)\Big).$$

Let S_g be the set of even characteristics, that is, $\varepsilon_1^t\varepsilon_2\equiv 0 \pmod 2$. For $g\geqslant 2$ and $\tau\in\mathbb{H}_g$, we put $h=|S_g|/2=2^{g-2}(2^g+1)$ and

$$\widetilde{\chi}_h(\tau)=\frac{(-1)^{gh/2}}{2^{2^{g-1}(2^g-1)}}\cdot\prod_{\boldsymbol{\varepsilon}\in S_g}\theta[\boldsymbol{\varepsilon}](\tau).$$

In [13], Igusa proves that if $g\geqslant 3$, then $\widetilde{\chi}_h\in\mathbf{R}_{g,h}(\mathbb{C})$. Starting from the analytic Siegel modular form $\widetilde{\chi}_h$, we define, thanks to Prop. 1.1.2, a geometric Siegel modular form

$$\chi_h(A_\tau)=(2i\pi)^{gh}\cdot\widetilde{\chi}_h(\tau)(dz_1\wedge\cdots\wedge dz_g)^{\otimes h}\in\mathbf{S}_{g,h}(\mathbb{C}).$$

Ichikawa proved several important results on this modular form that we summarize in the following proposition, see [11, Prop. 3.4] and [12]:

Proposition 1.1.3. *The geometric Siegel modular form χ_h belongs to $\mathbf{S}_{g,h}(\mathbb{Z})$. Moreover, there exists a Teichmüller modular form $\mu_{h/2} \in \mathbf{T}_{g,h/2}(\mathbb{Z})$ such that*

$$t^*(\chi_h) = (\mu_{h/2})^2. \qquad \square \; (2)$$

1.2. Action of isomorphisms

Let k be any field and $\phi : (A', a') \to (A, a)$ a $\bar{k}$-isomorphism of principally polarized abelian varieties. Choose a basis $(\omega_1, \ldots, \omega_g)$ of $\Omega^1_{\bar{k}}[A]$ and put $\omega = \omega_1 \wedge \ldots \wedge \omega_g \in \omega_k[A]$. If $\gamma_i = \phi^*(\omega_i)$, then $(\gamma_1, \ldots, \gamma_g)$ is a basis of $\Omega^1{}_{\bar{k}}[A']$ and by invariance under $\bar{k}$-isomorphisms

$$f((A, a), \omega) = f((A', a'), \gamma) \quad \text{where } \gamma = \gamma_1 \wedge \ldots \wedge \gamma_g \in \omega_{\bar{k}}[A'].$$

If $(\omega'_1, \ldots, \omega'_g)$ is another basis of $\Omega^1{}_{\bar{k}}[A']$ and $\omega' = \omega'_1 \wedge \ldots \wedge \omega'_g$, we denote by $M_\phi \in \mathrm{GL}_g(\bar{k})$ the matrix of the basis (γ_i) in the basis (ω'_i). Then one proves easily:

Proposition 1.2.1. *In the above notation,*

$$f((A, a), \omega) = \det(M_\phi)^h \cdot f((A', a'), \omega'). \qquad \square$$

First of all, from this formula applied to the action of -1, we deduce that, if k is a field of characteristic different from 2, then $\mathbf{S}_{g,h}(k) = \{0\}$ if gh is odd. From now on we assume that gh is even and $\mathrm{char}\, k \neq 2$.

Corollary 1.2.2. *Let (A, a) be a principally polarized abelian variety of dimension g defined over k and $f \in \mathbf{S}_{g,h}(k)$. Let $\omega_1, \ldots, \omega_g$ be a basis of $\Omega^1_k[A]$, and let $\omega = \omega_1 \wedge \ldots \wedge \omega_g \in \omega_k[A]$. Then*

$$\bar{f}(A, a) = f((A, a), \omega) \bmod^\times k^{\times h} \in k/k^{\times h}$$

does not depend on the choice of the basis of $\Omega^1_k[A]$. In particular $\bar{f}(A, a)$ is an invariant of the k-isomorphism class of (A, a). $\square$

Corollary 1.2.3. *Assume g odd. Let $f \in \mathbf{S}_{g,h}(k)$ and $\phi : (A', a') \to$ $\to (A, a)$ be a non trivial quadratic twist. There exists $c \in k \setminus k^2$ such that $\bar{f}(A, a) = c^{h/2} \bar{f}(A', a')$. Thus, if $\bar{f}(A, a) \neq 0$ then $\bar{f}(A, a)$ and $\bar{f}(A', a')$ do not belong to the same class in $k^\times / k^{\times h}$.*

Proof. Assume that ϕ is given by a quadratic character ε of $\mathrm{Gal}(\bar{k}/k)$. Then

$$d^\sigma = \varepsilon(\sigma)^g \cdot d, \text{ where } \quad d = \det(M_\phi) \in \bar{k}, \quad \sigma \in \mathrm{Gal}(\bar{k}/k).$$

Since g is odd, by our assumption, h is even. Moreover $d^2 = \varepsilon(\sigma) d d^\sigma \in k$. But $d \notin k$ since there exists σ such that $\varepsilon(\sigma) = -1$. Using Prop. 1.2.1 we find that

$$f((A, a), \omega) = (d^2)^{h/2} f((A', a'), \omega').$$

Since d^2 is not a square in k, if $\bar{f}(A,a) \neq 0$ then $\bar{f}(A,a)$ and $\bar{f}(A',a')$ belong to two different classes. $\square$

Let (A,a) be a principally polarized complex abelian variety of dimension g defined over $k \subset \mathbb{C}$. The *period matrix* of (A,a) defined by a basis $\omega_1, \ldots, \omega_g$ of $\Omega^1_k[A]$ and a symplectic basis $\gamma_1, \ldots, \gamma_{2g}$ of $H_1(A,\mathbb{Z})$ for the polarization a, is the Riemann matrix

$$\Omega = [\Omega_1 \ \Omega_2] = \begin{pmatrix} \int_{\gamma_1} \omega_1 & \cdots & \int_{\gamma_{2g}} \omega_1 \\ \vdots & & \vdots \\ \int_{\gamma_1} \omega_g & \cdots & \int_{\gamma_{2g}} \omega_g \end{pmatrix}.$$

One puts $\tau := \Omega_2^{-1}\Omega_1 \in \mathbb{H}_3$ in such a way that (A,a) is $\mathbb{C}$-isomorphic to A_τ. If C is a complex curve of genus g, one uses the same notation for the *period matrix* of C defined by a basis $\omega_1, \ldots, \omega_g$ of $\Omega^1_k[C]$, and a symplectic basis $\gamma_1, \ldots \gamma_{2g}$ of $H_1(C,\mathbb{Z})$ for the intersection pairing. By the canonical identifications

$$\Omega^1[C] = \Omega^1[\operatorname{Jac} C], \quad H_1(C,\mathbb{Z}) = H_1(\operatorname{Jac} C, \mathbb{Z}),$$

the period matrix of C is also the period matrix of $(\operatorname{Jac} C, j)$ defined by the corresponding bases. Applying Prop. 1.2.1 with the isomorphism $z \mapsto \Omega_2^{-1} z$, we get the following lemma.

Proposition 1.2.4. *In the above notation, let* $\omega = \omega_1 \wedge \ldots \wedge \omega_g \in$ $\in \omega_k[A]$. *Then*

$$f((A,a),\omega) = (2i\pi)^{gh} \frac{\widetilde{f}(\tau)}{\det \Omega_2^h}. \qquad \square$$

1.3. Jacobian among abelian threefolds

Serre stated in [16] and [20] the following precise form of Torelli's theorem:

Theorem 1.3.1. *Let* (A,a) *be a principally polarized abelian variety of dimension* $g \geqslant 1$ *over a field* k*, and assume that* (A,a) *is isomorphic over* $\bar{k}$ *to the Jacobian of a nonsingular curve* C*. Then* C *can be defined over* k*, and*

(i) *If* C *is hyperelliptic, there is an isomorphism, defined over* k*, from* (A,a) *to* $(\operatorname{Jac} C, j)$.

(ii) *If* C *is not hyperelliptic, there exists a quadratic character*

$$\varepsilon\colon \operatorname{Gal}(k^{\text{sep}}/k) \to \{\pm 1\}$$

and an isomorphism, defined over k, from the twisted abelian variety $(A,a)_\varepsilon$ *to* $(\operatorname{Jac}C, j)$. *Hence,* (A,a) *is k-isomorphic to a Jacobian if and only if* ε *is trivial.*

We restrict to the case where $k \subset \mathbb{C}$ and we now give a formula for ε. In order to do so, we need to recall some geometric results by Igusa. Denote by $\widetilde{\Sigma}_{140}$ the modular form defined by the thirty-fifth elementary symmetric function of the eighth power of the even Thetanullwerte. In his beautiful paper [13], Igusa proves the following result [*loc. cit.*, Lem. 10 and 11].

Theorem 1.3.2. *If* $\tau \in \mathbb{H}_3$, *then*:

(i) (A_τ, j) *is decomposable if* $\widetilde{\chi}_{18}(\tau) = \widetilde{\Sigma}_{140}(\tau) = 0$;
(ii) (A_τ, j) *is a hyperelliptic Jacobian if* $\widetilde{\chi}_{18}(\tau) = 0$ *and* $\widetilde{\Sigma}_{140}(\tau) \neq 0$;
(iii) (A_τ, j) *is a non hyperelliptic Jacobian if* $\widetilde{\chi}_{18}(\tau) \neq 0$. □

We are now able to prove our main result which can be seen as an arithmetic analogue of Igusa's result.

Theorem 1.3.3. *Let* (A,a) *be a principally polarized abelian threefold defined over* $k \subset \mathbb{C}$. *Let* $(\omega_1, \omega_2, \omega_3)$ *be any basis of* $\Omega^1_k[A]$ *and* $(\gamma_1, \dots, \gamma_6)$ *a symplectic basis of* $H_1(A, \mathbb{Z})$ *for the polarization a. Let* $\Omega = [\Omega_1\ \Omega_2]$ *be the period matrix defined by these bases, and* $\tau = \Omega_2^{-1}\Omega_1$.

(i) *If* $\widetilde{\Sigma}_{140}(\tau) = 0$ *and* $\widetilde{\chi}_{18}(\tau) = 0$ *then* (A,a) *is decomposable over* $\bar{k}$. *In particular it is not a Jacobian.*
(ii) *If* $\widetilde{\Sigma}_{140}(\tau) \neq 0$ *and* $\widetilde{\chi}_{18}(\tau) = 0$ *then there exists a hyperelliptic curve* C/k *such that* $(\operatorname{Jac}C, j) \simeq (A,a)$.
(iii) *If* $\widetilde{\chi}_{18}(\tau) \neq 0$ *then* (A,a) *is isomorphic to a non hyperelliptic Jacobian if and only if*

$$\chi_{18} := \chi_{18}((A,a),\omega) = (2i\pi)^{54} \frac{\widetilde{\chi}_{18}(\tau)}{\det \Omega_2^{18}}$$

is a square in k, with $\omega = \omega_1 \wedge \omega_2 \wedge \omega_3 \in \omega_k[A]$.

Proof. Only the third point is new. Indeed, the first and second points directly follow from Th. 1.3.2 and Th. 1.3.1. Suppose now that (A,a) is isomorphic over k to the Jacobian of a non hyperelliptic genus 3 curve C/k. Using successively Prop. 1.1.1 and Prop. 1.1.3, we get

$$\chi_{18}((A,a),\omega) = t^*(\chi_{18})(C,\lambda) = \mu_9(C,\lambda)^2 \in k^{\times 2},$$

with $\lambda = t^*\omega$. Hence, the desired expression is a square in $k^\times$. Its analytic expression on the right hand side of (iii) is a direct application of Prop. 1.2.4.

On the contrary, Cor. 1.2.3 shows that if (A, a) is a quadratic twist of a Jacobian (A', a') then there exists a non square $c \in k$ such that

$$\bar{\chi}_{18}(A, a) = c^9 \cdot \bar{\chi}_{18}(A', a').$$

As we have just proved that $\bar{\chi}_{18}(A', a')$ is a square in $k^\times/k^{\times 18}$, this implies that $\chi_{18}((A, a), \omega)$ is not. □

Corollary 1.3.4. *In the notation of Th. 1.3.3, the quadratic character* ε *of* $\mathrm{Gal}(\bar{k}/k)$ *introduced in Th. 1.3.1 is given by* $\varepsilon(\sigma) = d^\sigma/d$, *with* $d = \sqrt{\chi_{18}}$, *and with an arbitrary choice of the square root.* □

2. Invariants and Klein's formula

Let $d > 0$ be an integer. In this section k is an algebraically closed field of characteristic coprime with d.

2.1. Geometric invariants for nonsingular plane curves

We first review some classical invariant theory. Let E be a vector space of dimension n over k. The left regular representation r of $\mathrm{GL}(E)$ on the vector space $\mathsf{X}_d = \mathrm{Sym}^d(E^*)$ of forms of degree d on E is given by

$$r(u) : F(x) \mapsto (u \cdot F)(x) = F(ux)$$

for $u \in \mathrm{GL}(E)$, $F \in \mathsf{X}_d$ and $x \in E$. If U is an open subset of X_d stable under r, we still denote by r the left regular representation of $\mathrm{GL}(E)$ on the k-algebra $\mathscr{O}(U)$ of regular functions on U, in such a way that

$$r(u) : \Phi(F) \mapsto (u \cdot \Phi)(F) = \Phi(u \cdot F),$$

if $u \in \mathrm{GL}(E)$, $\Phi \in \mathscr{O}(U)$ and $F \in U$. If $h \in \mathbb{Z}$, we denote by $\mathscr{O}_h(U)$ the subspace, stable under r, of homogeneous elements of degree h. An element $\Phi \in \mathscr{O}_h(U)$ is an *invariant of degree h on U* if $u \cdot \Phi = \Phi$ for every $u \in \mathrm{SL}(E)$, and we denote by $\mathsf{Inv}_h(U)$ the subspace of invariants of degree h on U. Hence, if $\Phi \in \mathscr{O}(U)$, and if w and n are two integers such that $hd = nw$, then $\Phi \in \mathsf{Inv}_h(U)$ if and only if

$$u \cdot \Phi = (\det u)^w \Phi \quad \text{for every } u \in \mathrm{GL}(E),$$

and we call w the *weight* of Φ. Let $F \in \mathsf{X}_d$, and denote by $q_1, \ldots, q_n$ the partial derivatives of F. The *discriminant* of F is

$$\mathrm{Disc}\, F = c_{n,d}^{-1} \mathrm{Res}(q_1, \ldots, q_n), \quad \text{with} \quad c_{n,d} = d^{((d-1)^n - (-1)^n)/d},$$

where $\mathrm{Res}(q_1, \ldots, q_n)$ is the *multivariate resultant* of the forms $q_1, \ldots q_n$ [6, p. 426], the coefficient $c_{n,d}$ being chosen according to [20]. We refer to [15] for a detailed study of the discriminant of a ternary form, and the computation of the discriminant of a Ciani quartic.

From now on we assume $\dim E = n = 3$. The *universal curve* over X_d is the variety

$$\mathsf{Y}_d = \left\{(F,x) \in \mathsf{X}_d \times \mathbb{P}^2 \mid F(x) = 0\right\}.$$

The *nonsingular locus* of X_d is the principal open set

$$\mathsf{X}_d^0 = (\mathsf{X}_d)_{\mathrm{Disc}} = \left\{F \in \mathsf{X}_d \mid \mathrm{Disc}(F) \neq 0\right\}.$$

If Y_d^0 is the universal curve over the nonsingular locus X_d^0, the projection is a smooth surjective k-morphism

$$\pi\colon \mathsf{Y}_d^0 \to \mathsf{X}_d^0$$

whose fibre over F is the non singular plane curve C_F. If $F \in \mathsf{X}_d^0(k)$, we recall the usual way to write down explicitly the *classical basis* of $\Omega_k^1[C_F] = H^0(C_F, \Omega_{C_F}^1 \otimes k)$, see [2, p. 630]. Let

$$\eta^{(1)} = \frac{f(x_2dx_3 - x_3dx_2)}{q_1}, \quad \eta^{(2)} = \frac{f(x_3dx_1 - x_1dx_3)}{q_2},$$
$$\eta^{(3)} = \frac{f(x_1dx_2 - x_2dx_1)}{q_3},$$

where q_1, q_2, q_3 are the partial derivatives of F, and where $f \in \mathsf{X}_{d-3}$. The forms $\eta^{(i)}$ glue together and define a regular differential form $\eta_f(F) \in$ $\in \Omega_k^1[C_F]$. Since $\dim \mathsf{X}_{d-3} = (d-1)(d-2)/2 = g$, the linear map $f \mapsto$ $\mapsto \eta_f(F)$ defines an isomorphism

$$\mathsf{X}_{d-3} \xrightarrow{\sim} \Omega_k^1[C_F].$$

We denote $\eta_1, \ldots, \eta_g$ the sequence of sections obtained by substituting for f in η_f the g members of the canonical basis of X_{d-3}, enumerated according to the lexicographic order. Then $\eta = \eta_1 \wedge \ldots \wedge \eta_g$ is a section of

$$\boldsymbol{\alpha} = \bigwedge^g \pi_* \Omega^1_{\mathsf{Y}_d^0/\mathsf{X}_d^0},$$

the Hodge bundle on X_d^0. The map $u\colon x \mapsto ux$ induces an isomorphism

$$u\colon C_{u\cdot F} \xrightarrow{\sim} C_F$$

Hence, it has a natural action $u^*\colon \Omega_k^1[C_F] \to \Omega_k^1[C_{u\cdot F}]$ on the differentials and therefore, on the sections of $\boldsymbol{\alpha}^h$, for $h \in \mathbb{Z}$. More specifically, if $s\in\Gamma(\mathsf{X}_d^0,\boldsymbol{\alpha}^{\otimes h})$, one can write $s=\Phi\cdot\eta^{\otimes h}$ with $\Phi\in\mathscr{O}(\mathsf{X}_d^0)$; for $F\in\mathsf{X}_d^0$, one has

$$u^*s(F)=\Phi(F)\cdot(u^*\eta(F))^{\otimes h}.$$

The proof of the following lemma is left to the reader.

Lemma 2.1.1. *For any $u\in G$ and any $F\in\mathsf{X}_d^0$, the section $\eta\in\Gamma(\mathsf{X}_d^0,\boldsymbol{\alpha})$ satisfies,*

$$u^*\eta(F)=\det(u)^{w_0}\cdot\eta(u\cdot F),\quad \textit{with } w_0=\binom{d}{3}=\frac{dg}{3}\in\mathbb{N}.\quad\square$$

For any $h\in\mathbb{Z}$, we denote by $\Gamma(\mathsf{X}_d^0,\boldsymbol{\alpha}^{\otimes h})^G$ the subspace of sections $s\in\Gamma(\mathsf{X}_d^0,\boldsymbol{\alpha}^{\otimes h})$ such that $u^*s(F)=s(u\cdot F)$ for every $u\in G$ and $F\in\mathsf{X}_d^0$.

Proposition 2.1.2. *Let $h\geqslant 0$ be an integer. The linear map*

$$\Phi\mapsto\rho(\Phi)=\Phi\cdot\eta^{\otimes h}$$

is an isomorphism

$$\rho:\mathsf{Inv}_{gh}(\mathsf{X}_d^0)\xrightarrow{\sim}\Gamma(\mathsf{X}_d^0,\boldsymbol{\alpha}^{\otimes h})^G.$$

Proof. Let $\Phi\in\mathsf{Inv}_{gh}(\mathsf{X}_d^0)$, $s=\rho(\Phi)=\Phi\cdot\eta^{\otimes h}$, and $w=dgh/3$, the weight of Φ. Then using Lem. 2.1.1,

$$\begin{aligned}u^*s(F)&=\Phi(F)\cdot(u^*\eta(F))^{\otimes h}=\Phi(F)\cdot\det(u)^{w_0h}\cdot\eta(u\cdot F)^{\otimes h}\\&=\det(u)^w\Phi(F)\cdot\eta(u\cdot F)^{\otimes h}=\Phi(u\cdot F)\cdot\eta(u\cdot F)^{\otimes h}=s(u\cdot F).\end{aligned}$$

Hence, $\rho(\Phi)\in\Gamma(\mathsf{X}_d^0,\boldsymbol{\lambda}^{\otimes h})^G$. Conversely, the inverse of ρ is the map $s\mapsto s/\eta^{\otimes h}$, and this proves the proposition. $\square$

2.2. Modular forms as invariants

Let $d>2$ be an integer and $g=\binom{d-1}{2}$. Since the fibres of $\mathsf{Y}_d^0\to\mathsf{X}_d^0$ are nonsingular non hyperelliptic plane curves of genus g, by the universal property of M_g we get a morphism

$$p:\mathsf{X}_g^0\;\to\;\mathsf{M}_g.$$

and $p^*\boldsymbol{\lambda}=\boldsymbol{\alpha}$ by construction. This induces a linear map

$$p^*:\mathbf{T}_{g,h}(k)\;\longrightarrow\;\Gamma(\mathsf{X}_d^0,\boldsymbol{\alpha}^{\otimes h}).$$

Moreover, for $u\in G$, since $u\colon C_{u\cdot F}\to C_F$ is an isomorphism, we get the following commutative diagram

$$\begin{tikzcd}\boldsymbol{\lambda}[C_F] \arrow[r, "u^*"] \arrow[d, "p^*"'] & \boldsymbol{\lambda}[C_{u\cdot F}] \arrow[d, "p^*"] \\ \boldsymbol{\alpha}[F] \arrow[r, "u^*"] & \boldsymbol{\alpha}[u\cdot F].\end{tikzcd}$$

For any $f \in \mathbf{T}_{g,h}(k)$, the modular invariance of f means that

$$u^* f(C_F) = f(C_{u \cdot F}).$$

Then

$$u^*[(p^* f)(F)] = u^*[p^*(f(C_F))] = p^*[u^* f(C_F)] = \\ = p^*[f(C_{u \cdot F})] = (p^* f)(u \cdot F),$$

and this means that $p^* f \in \Gamma(\mathsf{X}_d^0, \boldsymbol{\alpha}^{\otimes h})^G$. If $g = 3$ then p^* is a linear isomorphism. Combining this result with Prop. 2.1.2, we obtain:

Proposition 2.2.1. *For any integer $h \geqslant 0$, the linear map $\sigma = \rho^{-1} \circ p^*$ is a homomorphism:*

$$\mathbf{T}_{g,h}(k) \;\rightarrow\; \mathsf{Inv}_{gh}(\mathsf{X}_d^0)$$

such that

$$\sigma(f)(F) = f(C_F, \lambda)$$

with $\lambda = (p^*)^{-1}\eta$, for any $F \in \mathsf{X}_d^0$ and any section $f \in \mathbf{T}_{g,h}(k)$. If $g = 3$, then σ is an isomorphism. □

We finally make a link between invariants and analytic Siegel modular forms. Let $F \in X_d^0(\mathbb{C})$ and $(\eta_1, \ldots, \eta_g)$ the basis of regular differentials on C_F defined in § 2.1. Let $(\gamma_1, \ldots \gamma_{2g})$ be a symplectic basis of $H_1(C, \mathbb{Z})$ for the intersection pairing. Let $\Omega = [\Omega_1 \;\; \Omega_2]$ the period matrix of C_F defined by these bases, and $\tau = \Omega_2^{-1}\Omega_1$.

Corollary 2.2.2. *Let $f \in \mathbf{S}_{g,h}(\mathbb{C})$ be a geometric Siegel modular form, $\widetilde{f} \in \mathbf{R}_{g,h}(\mathbb{C})$ the corresponding analytic modular form, and $\Phi = \sigma(t^* f)$ the corresponding invariant. In the above notation,*

$$\Phi(F) = (2i\pi)^{gh} \frac{\widetilde{f}(\tau)}{\det \Omega_2^h}.$$

Proof. Let $\lambda = (p^*)^{-1}(\eta)$ and $\omega = (t^*)^{-1}(\lambda)$. From Prop. 1.1.1 and 2.2.1, we deduce

$$\Phi(F) = (t^* f)(C_F, \lambda) = f(\mathrm{Jac}\, C_F, \omega),$$

and Prop. 1.2.4 give the result, since Ω is also the period matrix of $\mathrm{Jac}\, C_F$. □

We are now ready to give a proof of the following result [14, Eq. 118, p. 462]:

Theorem 2.2.3 (Klein's formula)**.** *Let $F \in \mathsf{X}_4^0(\mathbb{C})$ and C_F be the corresponding smooth plane quartic. Let (η_1, η_2, η_3) be the classical basis of $\Omega^1_{\mathbb{C}}[C_F]$ and $(\gamma_1, \ldots, \gamma_6)$ be a symplectic basis of $H_1(C_F, \mathbb{Z})$ for the inter-*

section pairing. Let $\Omega = [\Omega_1\ \Omega_2]$ *the period matrix of* C_F *defined by these bases, and* $\tau = \Omega_2^{-1}\Omega_1$. *Then*

$$\mathrm{Disc}(F)^2 = (2\pi)^{54}\frac{\widetilde{\chi}_{18}(\tau)}{\det(\Omega_2)^{18}}.$$

Proof. Cor. 2.2.2 shows that for any $F \in \mathsf{X}_4^0$ the invariant $I = \sigma \circ t^*(\chi_{18})$ satisfies

$$I(F) = (2i\pi)^{54}\frac{\widetilde{\chi}_{18}(\tau)}{\det \Omega_2^{18}}.$$

Moreover Th. 1.3.2 (iii) shows that $I(F) \neq 0$ for all $F \in \mathsf{X}_4^0$. Since the discriminant is an irreducible polynomial, as immediate consequence of Hilbert's Nullstellensatz we get that $I = c\,\mathrm{Disc}^n$ with $c \in \mathbb{C}^\times$ a constant and $n \in \mathbb{N}$. Since I is an invariant of weight 54 and Disc an invariant of weight 27, $n = 2$. Finally, it is proven in [15, Cor. 4.2] that Klein's formula holds true for any Ciani quartic with $c = 1$. □

Remark 2.2.4. Th. 2.2.3 implies that

$$\mu_9(C_F, \lambda) = \pm\,\mathrm{Disc}\,F.$$

This might be deduced from the definition of μ_9, although it seems that this fact was not observed before.

2.3. Beyond genus 3

First of all, note that an analogue of Klein's formula has been derived in the hyperelliptic case by Lockhart [17] and also by Guàrdia [8]. Their formula is a direct consequence of Thomae's formula [21]. Now, it is natural to try to extend the preceding results to the case $g > 3$. For Klein's formula and $g = 4$, Klein himself, in the footnote of p. 462 in [14], gives the amazing formula

$$\frac{\widetilde{\chi}_{68}(\tau)}{\det(\Omega_2)^{68}} = c \cdot \Delta(C)^2 \cdot T(C)^8. \tag{3}$$

Here $\tau = \Omega_2^{-1}\Omega_1$, with $\Omega = [\Omega_1\ \Omega_2]$ a period matrix of a genus 4 non hyperelliptic curve C given in $\mathbb{P}^3$ as an intersection of a quadric Q and a cubic surface E. The elements $\Delta(C)$ and $T(C)$ are defined in classical invariant theory as, respectively, the discriminant of Q and the tact invariant of Q and E (see [19, p. 122]). No such formula seems to be known in the non hyperelliptic case for $g > 4$.

Let us now look at what happens when we try to apply Serre's approach for $g > 3$. To begin with, when g is even, we cannot use Cor. 1.2.2

to distinguish between quadratic twists. Let us assume that g is odd. Cor. 1.2.3 shows that there exists $c \in k \setminus k^2$ such that

$$\bar{\chi}_h(A', a') = c^{h/2} \cdot \bar{\chi}_h(A, a)$$

for a Jacobian (A, a) and a quadratic twist (A', a'). What enabled us to distinguish between the two when $g = 3$ is that $h/2 = 9$ is odd. However as soon as $g > 3$, the 2-valuation of $h/2$ is $g - 3 > 0$, so it is not enough for $\bar{\chi}_h(A)$ to be a square in k to make a distinction between A and A'. It must rather be a 2^{g-2}-th power in k. It can be easily seen from the proof of [22, Th. 1] that $t^*(\chi_h)$ does not admit a fourth root. According to [1] or [23] this implies $\bar{\chi}_h(A, a)$ is not a 2^{g-2}-th power in k for infinitely many Jacobians (A, a) defined over number fields k. So we can no longer use the modular form χ_h to characterize Jacobians over k.

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On logarithmic derivatives of zeta functions in families of global fields

(with P. Lebacque)

The goal of this paper is to get a formula for the limit of logarithmic derivatives of zeta functions in families of global fields (assuming GRH in the number field case) with an explicit error term. This result is close in spirit both to the explicit Brauer—Siegel and Mertens theorems from [2] as well as to the generalized Brauer—Siegel type theorems from a paper by the first author. We also improve the error term in the explicit Brauer—Siegel theorem from [2], allowing its dependence on the family of global fields under consideration.

Throughout the paper the constants involved in O and $\ll$ are absolute and effective (and, in fact, not very large). Let K be a global field that is a finite extension of $\mathbb{Q}$ or a finite extension of $\mathbb{F}_r(t)$, in the latter case $K=\mathbb{F}_r(X)$ for a smooth absolutely irreducible projective curve over $\mathbb{F}_r$, where $\mathbb{F}_r$ is the finite field with r elements. We will often use the acronyms NF or FF for the statements proven in the number field and the function field cases respectively. We shall often omit the index K in our notation in cases when it creates no confusion.

For a number field K let n_K and D_K denote its degree and its discriminant respectively. Let g_K be the genus of a function field, that is the genus of the corresponding smooth projective curve and let $g_K=\log\sqrt{|D_K|}$ in the number field case. Let $\mathscr{P}(K)$ be the set of (finite) places of K and let $\Phi_q=\Phi_q(K)$ be the number of places of norm q in K, i. e.

$$\Phi_q = |\{\mathfrak{p}\in\mathscr{P}(K) \mid \mathrm{N}\mathfrak{p}=q\}|.$$

In the number field case we denote by $\Phi_{\mathbb{R}}=r_1$ and $\Phi_{\mathbb{C}}=r_2$ the number of real and (pairs of) complex places of K respectively.

Recall that the zeta function of a global field K may be defined as

$$\zeta_K(s)=\prod_q(1-q^{-s})^{-\Phi_q},$$

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where the product runs over all prime powers q. We denote by

$$Z_K(s) = -\sum_q \frac{\Phi_q \log q}{q^s - 1}$$

its logarithmic derivative. One knows that $\zeta_K(s)$ can be analytically continued to the whole complex plane and satisfies a functional equation relating $\zeta_K(s)$ and $\zeta_K(1-s)$. Furthermore, in the function field case $\zeta_K(s)$ is a rational function of $t = r^{-s}$. Moreover,

$$\zeta_K(s) = \frac{\prod_{j=1}^{g} (\pi_j t - 1)(\bar{\pi}_j t - 1)}{(1-t)(1-rt)},$$

and $|\pi_j| = \sqrt{r}$ (the Riemann hypothesis). For the rest of the paper we will assume that the Generalized Riemann Hypothesis is true for zeta functions of number fields, that is all the non-trivial zeroes of $\zeta_K(s)$ are on the line $\operatorname{Re} s = \frac{1}{2}$.

Here are our main results:

Theorem 1 (FF). *For any function field K, any integer $N \geqslant 10$ and any $\varepsilon = \varepsilon_0 + i\varepsilon_1$ such that $\varepsilon_0 = \operatorname{Re}\varepsilon > 0$ we have:*

$$\sum_{f=1}^{N} \frac{f \Phi_{r^f}}{r^{\left(\frac{1}{2}+\varepsilon\right)f} - 1} + \frac{1}{\log r} \cdot Z_K\left(\frac{1}{2} + \varepsilon\right) + \frac{1}{r^{-\frac{1}{2}+\varepsilon} - 1} = \\ = O\left(\frac{g_K}{r^{\varepsilon_0 N}}\left(1 + \frac{1}{\varepsilon_0}\right)\right) + O\left(r^{\frac{N}{2}}\right).$$

Theorem 2 (NF, GRH). *For a number field K, an integer $N \geqslant 10$ and any $\varepsilon = \varepsilon_0 + i\varepsilon_1$ such that $\varepsilon_0 = \operatorname{Re}\varepsilon > 0$ we have:*

$$\sum_{q \leqslant N} \frac{\Phi_q \log q}{q^{\frac{1}{2}+\varepsilon} - 1} + Z_K\left(\frac{1}{2} + \varepsilon\right) + \frac{1}{\varepsilon - \frac{1}{2}} = \\ = O\left(\frac{|\varepsilon|^4 + |\varepsilon|}{\varepsilon_0^2}(g_K + n \log N)\frac{\log^2 N}{N^{\varepsilon_0}}\right) + O\left(\sqrt{N}\right).$$

Let us explain a little bit the meaning of these theorems. It was known before (see below) that the identities (without the error terms) of the theorems are true in the asymptotic sense (when $N = \infty$ and $g = \infty$ for families of global fields). Our theorems give the "finite level" versions of these results. They allow to estimate how well the cutoffs of the series for $Z_K(s)$ approximate it away from the domain of convergence of this series (which is $\operatorname{Re} s > 1$) when we vary K.

The proofs of these theorems are based on the Weil explicit formula. However, in the number field case the analytical difficulties are rather considerable, so the explicit formula has to be applied three times with different choices of test functions. We note that in both cases we also obtain the new proofs of the basic inequalities from [3] and [5] (c.f. formulae (1) and (2) below).

Our next results concern families of global fields $\{K_i\}$ with growing genus $g_i = g(K_i)$. Recall ([4],[5]) that a family of global fields is called asymptotically exact if the limits

$$\phi_\alpha = \phi_\alpha(\{K_i\}) = \lim_{i\to\infty} \frac{\Phi_\alpha(K_i)}{g_i}$$

exist for each α which is a power of r in the function field case and each prime power and $\alpha = \mathbb{R}$ and $\alpha = \mathbb{C}$ in the number field case. The numbers ϕ_α are called the Tsfasman—Vlăduţ invariants of the family $\{K_i\}$. From now on we assume that all our families are asymptotically exact.

We introduce the limit zeta function of a family $\{K_i\}$ as

$$\zeta_{\{K_i\}}(s) = \prod_q (1-q^{-s})^{-\phi_q}.$$

We will also denote by $Z_{\{K_i\}}(s) = -\sum_q \frac{\phi_q \log q}{q^s - 1}$ its logarithmic derivative. The basic inequality (c.f. [3] and [5]) can be formulated as

$$\sum_{f=1}^{\infty} \frac{f\phi_{r^f}}{r^{\frac{f}{2}}-1} \leqslant 1 \tag{1}$$

in the function field case and as

$$\sum_q \frac{\phi_q \log q}{\sqrt{q}-1} + \phi_{\mathbb{R}}\Big(\log(2\sqrt{2\pi}) + \frac{\pi}{4} + \frac{\gamma}{2}\Big) + \phi_{\mathbb{C}}\Big(\log(8\pi)+\gamma\Big) \leqslant 1 \tag{2}$$

in the number field case. It follows from the inequality that both the product and the sum converge absolutely for $\operatorname{Re} s \geqslant \frac{1}{2}$ and thus define analytic functions for $\operatorname{Re} s > \frac{1}{2}$.

Let us first formulate a corollary of Theorems 1 and 2.

Corollary 1. *For an asymptotically exact family of global fields* $\{K_i\}$, *an integer* $N \geqslant 10$ *and any* $\varepsilon = \varepsilon_0 + i\varepsilon_1$ *such that* $\varepsilon_0 = \operatorname{Re}\varepsilon > 0$ *the following holds*:

1) *in the function field case*:

$$\sum_{f=1}^{N} \frac{f\phi_{r^f}}{r^{\left(\frac{1}{2}+\varepsilon\right)f}-1} + \frac{1}{\log r}\cdot Z_{\{K_i\}}\Big(\frac{1}{2}+\varepsilon\Big) = O\Big(\frac{1}{r^{\varepsilon_0 N}}\Big(1+\frac{1}{\varepsilon_0}\Big)\Big);$$

2) *in the number field case with the assumption of GRH*:

$$\sum_{q\leqslant N}\frac{\phi_q\log q}{q^{\frac{1}{2}+\varepsilon}-1}+Z_{\{K_i\}}\Big(\frac{1}{2}+\varepsilon\Big)=O\Big(\frac{(|\varepsilon|^4+|\varepsilon|)\log^3 N}{\varepsilon_0^2N^{\varepsilon_0}}\Big).$$

This corollary, in particular, implies the convergence of the logarithmic derivatives of zeta functions of global fields to the logarithmic derivative of the limit zeta function for $\operatorname{Re} s>\frac{1}{2}$. This result (without an explicit error term but with a much easier proof) has been recently obtained in paper by the first of the authors in the function field case.

Our next result concerns the behaviour of $Z_{\{K_i\}}(s)$ at $s=\frac{1}{2}$.

Theorem 3. *For an asymptotically exact family of global fields $\{K_i\}$ there exists a number $\delta>0$ depending on $\{K_i\}$ such that:*

1) *in the function field case:*

$$\sum_{f=1}^{N}\frac{f\phi_{r^f}}{r^{\frac{f}{2}}-1}+\frac{1}{\log r}\cdot Z_{\{K_i\}}\Big(\frac{1}{2}\Big)=O(r^{-\delta N});$$

2) *in the number field case, assuming GRH, we have:*

$$\sum_{q\leqslant N}\frac{\phi_q\log q}{\sqrt{q}-1}+Z_{\{K_i\}}\Big(\frac{1}{2}\Big)=O(N^{-\delta}).$$

Let us formulate a corollary of this result which, in a sense, improves the explicit Brauer—Siegel theorem from [2]. We denote by $\varkappa_{K_i}=\operatorname{Res}_{s=1}\zeta_{K_i}(s)$ the residue of $\zeta_{K_i}(s)$ at $s=1$. We let $\kappa=\kappa_{\{K_i\}}=\lim\limits_{i\to\infty}\frac{\log\varkappa_{K_i}}{g_i}$. One knows ([4] and [5]) that for an asymptotically exact family this limit exists and equals $\log\zeta_{\{K_i\}}(1)$ (we assume GRH in the number field case). In fact, in the number field case it can be seen as a generalization of the classical Brauer—Siegel theorem (cf. [1]).

Corollary 2. *For an asymptotically exact family of global fields $\{K_i\}$ there exists a number $\delta>0$ depending on $\{K_i\}$ such that*:

1) *in the function field case*:

$$\sum_{f=1}^{N}\phi_{r^f}\log\frac{r^f}{r^f-1}=\kappa+O\Big(\frac{1}{r^{(\frac{1}{2}+\delta)N}N}\Big);$$

2) *assuming GRH, in the number field case*:

$$\sum_{q\leqslant N}\phi_q\log\frac{q}{q-1}=\kappa+O\Big(\frac{1}{N^{\frac{1}{2}+\delta}\log N}\Big).$$

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On logarithmic derivatives of zeta functions in families of global fields

(with P. Lebacque)

Abstract. We prove a formula for the limit of logarithmic derivatives of zeta functions in families of global fields with an explicit error term. This can be regarded as a rather far reaching generalization of the explicit Brauer—Siegel theorem both for number fields and function fields.

1. Introduction

The goal of this paper is to prove a formula for the limit of logarithmic derivatives of zeta functions in families of global fields (assuming GRH in the number field case) with an explicit error term. This result is close in spirit both to the explicit Brauer—Siegel and Mertens theorems from [9] and to the asymptotic theorem for Dedekind zeta functions from [17]. We also improve the error term in the explicit Brauer—Siegel theorem from [9], allowing its dependence on the family of global fields under consideration.

Throughout the paper the constants involved in O and $\ll$ are absolute and effective (and, in fact, not very large). Let K be a global field that is a finite extension of $\mathbb{Q}$ or a finite extension of $\mathbb{F}_r(t)$, in the latter case $K = \mathbb{F}_r(X)$ for a smooth absolutely irreducible projective curve over $\mathbb{F}_r$, where $\mathbb{F}_r$ is the finite field with r elements. We will often use the acronyms NF or FF for the statements proven in the number field and the function field cases respectively. We shall often omit the index K in our notation in cases when it creates no confusion.

For a number field K let n_K and D_K denote its degree and its discriminant respectively. Let g_K be the genus of a function field, that is the genus of the corresponding smooth projective curve and let $g_K = \log \sqrt{|D_K|}$ in

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the number field case. Let $\mathscr{P}(K)$ be the set of (finite) places of K and let $\Phi_q = \Phi_q(K)$ be the number of places of norm q in K, i. e.

$$\Phi_q = |\{\mathfrak{p} \in \mathscr{P}(K) \mid \mathrm{N}\mathfrak{p} = q\}|.$$

In the number field case we denote by $\Phi_{\mathbb{R}} = r_1$ and $\Phi_{\mathbb{C}} = r_2$ the number of real and complex places of K respectively.

Recall that the zeta function of a global field K is defined as

$$\zeta_K(s) = \prod_q (1 - q^{-s})^{-\Phi_q},$$

where the product runs over all prime powers q. We denote by

$$Z_K(s) = -\sum_q \frac{\Phi_q \log q}{q^s - 1}$$

its logarithmic derivative. One knows that $\zeta_K(s)$ can be analytically continued to the whole complex plane and satisfies a functional equation relating $\zeta_K(s)$ and $\zeta_K(1-s)$. Furthermore, in the function field case $\zeta_K(s)$ is a rational function of $t = r^{-s}$. Moreover,

$$\zeta_K(s) = \frac{\prod_{j=1}^{g} (\pi_j t - 1)(\bar{\pi}_j t - 1)}{(1-t)(1-rt)}, \tag{1.1}$$

and $|\pi_j| = \sqrt{r}$ (the Riemann hypothesis). For the rest of the paper we will assume that the Generalized Riemann Hypothesis is true for zeta functions of number fields, that is all the non-trivial zeroes of $\zeta_K(s)$ are on the line $\mathrm{Re}\, s = \frac{1}{2}$.

Here are our first main results:

Theorem 1.1 (FF). *For any function field K, any integer $N \geqslant 10$ and any $\varepsilon = \varepsilon_0 + i\varepsilon_1$ such that $\varepsilon_0 = \mathrm{Re}\,\varepsilon > 0$ we have*:

$$\sum_{f=1}^{N} \frac{f\Phi_{r^f}}{r^{\left(\frac{1}{2}+\varepsilon\right)f} - 1} + \frac{1}{\log r} \cdot Z_K\Big(\frac{1}{2} + \varepsilon\Big) + \frac{r^{n\left(\frac{1}{2}-\varepsilon\right)}}{r^{\varepsilon - \frac{1}{2}} - 1} = O\Big(\frac{g_K}{r^{\varepsilon_0 N}}\Big(1 + \frac{1}{\varepsilon_0}\Big)\Big).$$

Theorem 1.2 (NF, GRH). *For a number field K, an integer $N \geqslant 10$ and any $\varepsilon = \varepsilon_0 + i\varepsilon_1$ such that $\varepsilon_0 = \mathrm{Re}\,\varepsilon > 0$ we have*:

$$\sum_{q \leqslant N} \frac{\Phi_q \log q}{q^{\frac{1}{2}+\varepsilon} - 1} + Z_K\Big(\frac{1}{2} + \varepsilon\Big) + \frac{\left(N + \frac{1}{2}\right)^{\frac{1}{2}-\varepsilon}}{\varepsilon - \frac{1}{2}} =$$

$$= O\Big(\frac{|\varepsilon|^4 + |\varepsilon|}{\varepsilon_0^2}(g + n \log N)\frac{\log^2 N}{N^{\varepsilon_0}}\Big).$$

Let us explain a little bit the meaning of these theorems. It was known before (see [17] and also below) that the identities (without the error terms) of the theorems are true in the asymptotic sense (when $N=\infty$ and $g=\infty$ for families of global fields). Our theorems give the "finite level" versions of these results. They allow to estimate how well the cutoffs of the series for $Z_K(s)$ approximate it away from the domain of convergence of this series (which is $\operatorname{Re} s>1$) when we vary K.

We give the proof of these theorems in Sections 2 and 3 respectively. Both proofs are based on the Weil explicit formula. However, in the number field case the analytical difficulties are rather considerable, so the explicit formula has to be applied three times with different choices of test functions. We note that, as indicated in the remarks in the corresponding sections, in both cases we obtain the new proofs of the basic inequalities from [14] and [16].

Our next results concern families of global fields $\{K_i\}$ with growing genus $g_i=g(K_i)$. Recall [15, 16] that a family of global fields is called asymptotically exact if the limits

$$\phi_\alpha=\phi_\alpha(\{K_i\})=\lim_{i\to\infty}\frac{\Phi_\alpha(K_i)}{g_i}$$

exist for each α which is a power of r in the function field case and each prime power and $\alpha=\mathbb{R}$ and $\alpha=\mathbb{C}$ in the number field case. The numbers ϕ_α are called the Tsfasman—Vlăduţ invariants of the family $\{K_i\}$. From now on we assume that all our families are asymptotically exact.

We introduce the limit zeta function of a family $\{K_i\}$ as

$$\zeta_{\{K_i\}}(s)=\prod_q(1-q^{-s})^{-\phi_q}.$$

We will also denote by $Z_{\{K_i\}}(s)=-\sum\limits_q\dfrac{\phi_q\log q}{q^s-1}$ its logarithmic derivative. It follows from the basic inequality (cf. [14] and [16] or Sections 2 and 3 of this paper) that both the product and the sum converge absolutely for $\operatorname{Re} s\geqslant\dfrac12$ and thus define analytic functions for $\operatorname{Re} s>\dfrac12$.

Let us first formulate a corollary of Theorems 1.1 and 1.2.

Corollary 1.3. *For an asymptotically exact family of global fields $\{K_i\}$, an integer $N\geqslant10$ and any $\varepsilon=\varepsilon_0+i\varepsilon_1$ such that $\varepsilon_0=\operatorname{Re}\varepsilon>0$ the following holds:*

1) *in the function field case:*

$$\sum_{f=1}^N\frac{f\phi_{r^f}}{r^{\left(\frac12+\varepsilon\right)f}-1}+\frac{1}{\log r}\cdot Z_{\{K_i\}}\Big(\frac12+\varepsilon\Big)=O\Big(\frac{1}{r^{\varepsilon_0N}}\Big(1+\frac{1}{\varepsilon_0}\Big)\Big);$$

2) *in the number field case with the assumption of GRH:*

$$\sum_{q\leqslant N}\frac{\phi_q\log q}{q^{\frac{1}{2}+\varepsilon}-1}+Z_{\{K_i\}}\Big(\frac{1}{2}+\varepsilon\Big)=O\Big(\frac{(|\varepsilon|^4+|\varepsilon|)\log^3 N}{\varepsilon_0^2N^{\varepsilon_0}}\Big).$$

This corollary, in particular, implies the convergence of the logarithmic derivatives of zeta functions of global fields to the logarithmic derivative of the limit zeta function for $\operatorname{Re} s>\frac{1}{2}$. This result (without an explicit error term but with a much easier proof) has been recently obtained in [17].

Our next result concerns the behaviour of $Z_{\{K_i\}}(s)$ at $s=\frac{1}{2}$.

Theorem 1.4. *For an asymptotically exact family of global fields* $\{K_i\}$ *there exists a number* $\delta>0$ *depending on* $\{K_i\}$ *such that*:

1) *in the function field case*:

$$\sum_{f=1}^{N}\frac{f\phi_{r^f}}{r^{\frac{f}{2}}-1}+\frac{1}{\log r}\cdot Z_{\{K_i\}}\Big(\frac{1}{2}\Big)=O(r^{-\delta N});$$

2) *in the number field case, assuming GRH, we have*:

$$\sum_{q\leqslant N}\frac{\phi_q\log q}{\sqrt{q}-1}+Z_{\{K_i\}}\Big(\frac{1}{2}\Big)=O(N^{-\delta}).$$

Let us formulate a corollary of this result which, in a sense, improves the explicit Brauer—Siegel theorem from [9]. We denote by $\varkappa_{K_i}=\operatorname{Res}_{s=1}\zeta_{K_i}(s)$ the residue of $\zeta_{K_i}(s)$ at $s=1$. We let $\kappa=\kappa_{\{K_i\}}=\lim\limits_{i\to\infty}\frac{\log\varkappa_{K_i}}{g_i}$. One knows ([15] and [16]) that for an asymptotically exact family this limit exists and equals $\log\zeta_{\{K_i\}}(1)$ (we assume GRH in the number field case). In fact, in the number field case it can be seen as a generalization of the classical Brauer—Siegel theorem (cf. [7]).

Corollary 1.5. *For an asymptotically exact family of global fields* $\{K_i\}$ *there exists a number* $\delta>0$ *depending on* $\{K_i\}$ *such that*:

1) *in the function field case:*

$$\sum_{f=1}^{N}\phi_{r^f}\log\frac{r^f}{r^f-1}=\kappa+O\Big(\frac{1}{r^{(\frac{1}{2}+\delta)N}N}\Big);$$

2) *assuming GRH, in the number field case:*

$$\sum_{q\leqslant N}\phi_q\log\frac{q}{q-1}=\kappa+O\Big(\frac{1}{N^{\frac{1}{2}+\delta}\log N}\Big).$$

We prove Theorem 1.4 and both of the Corollaries 1.3 and 1.5 in the § 4.

2. Proof of Theorem 1.1

We will use the following analogue of Weil explicit formula for zeta functions of function fields, see [12] or [5] (in the case of varieties over finite fields) for a proof.

Theorem 2.1. *For a sequence $v=(v_n)$ such that $\sum_{n=1}^{\infty} v_n r^{\frac{n}{2}}$ is convergent, the series $\sum_{n=1}^{\infty} v_n r^{-\frac{n}{2}} \sum_{m|n} m\Phi_{r^m}$ is also convergent and one has the following equality*:

$$\sum_{n=1}^{\infty} v_n r^{-\frac{n}{2}} \sum_{f|n} f\Phi_{r^f} = \psi_v(r^{1/2}) + \psi_v(r^{-1/2}) - \sum_{j=1}^{g}\left(\psi_v\left(\frac{\pi_j}{\sqrt{r}}\right) + \psi_v\left(\frac{\bar{\pi}_j}{\sqrt{r}}\right)\right)$$

where the π_j, $\bar{\pi}_j$ are the inverse roots of the numerator of the zeta function of K, $g=g_K$ and $\psi_v(t)=\sum_{n=1}^{\infty} v_n t^n$.

Let us take the test sequence $v_n = v_n(N) = \frac{1}{r^{n\varepsilon}}$ if $n \leqslant N$ and 0 otherwise. Introducing it in the explicit formulae, we get

$$S_0(N,\varepsilon) = S_1(N,\varepsilon) + S_2(N,\varepsilon) - S_3(N,\varepsilon),$$

where

$$S_0(N,\varepsilon) = \sum_{n=1}^{N} r^{-n\left(\frac{1}{2}+\varepsilon\right)} \sum_{f|n} f\Phi_{r^f}, \quad S_1(N,\varepsilon) = \sum_{n=1}^{N} r^{n\left(\frac{1}{2}-\varepsilon\right)},$$

$$S_2(N,\varepsilon) = \sum_{n=1}^{N} r^{-n\left(\frac{1}{2}+\varepsilon\right)}, \qquad S_3(N,\varepsilon) = \sum_{j=1}^{g}\sum_{n=1}^{N} r^{-n\left(\frac{1}{2}+\varepsilon\right)}(\pi_j^n + \bar{\pi}_j^n).$$

Let us estimate each of the S_i.

Calculation of S_0:

Let us first change the summation order in S_0:

$$S_0(N,\varepsilon) = \sum_{n=1}^{N} r^{-n\left(\frac{1}{2}+\varepsilon\right)} \sum_{f|n} f\Phi_{r^f} = \sum_{f=1}^{N} f\Phi_{r^f} \sum_{m=1}^{[N/f]} \frac{1}{r^{fm\left(\frac{1}{2}+\varepsilon\right)}}.$$

Now

$$R_0(N,\varepsilon) = \sum_{f=1}^{N} f\Phi_{r^f} \frac{1}{r^{\left(\frac{1}{2}+\varepsilon\right)f} - 1} - S_0(N,\varepsilon) =$$

$$= \sum_{f=1}^{N} f\Phi_{r^f}\left(\frac{1}{r^{\left(\frac{1}{2}+\varepsilon\right)f}-1} - \sum_{m=1}^{[N/f]} r^{-fm\left(\frac{1}{2}+\varepsilon\right)}\right) = \sum_{f=1}^{N} f\Phi_{r^f} \sum_{m=[N/f]+1}^{\infty} r^{-fm\left(\frac{1}{2}+\varepsilon\right)}.$$

Taking the absolute values, we can assume that ε is real. Summing the geometric series, we obtain

$$0 \leqslant \sum_{f=1}^{N} f\Phi_{r^f} \frac{1}{r^{\left(\frac{1}{2}+\varepsilon\right)f}-1} - S_0(N,\varepsilon) \leqslant \sum_{f=1}^{N} f\Phi_{r^f} r^{-\left(\frac{1}{2}+\varepsilon\right)[N/f]f} \frac{1}{r^{\left(\frac{1}{2}+\varepsilon\right)f}-1}.$$

We now use the Weil inequality $f\Phi_{r^f} \leqslant r^f+1+2g\sqrt{r^f}$, and split the above sum into two parts in the following way. For $f>[N/2]$ we have $[N/f]=1$ and for $f \leqslant [N/2]$ we use the inequality $f[N/f] \geqslant N-f$.

$$|R_0(N,\varepsilon)| \leqslant \sum_{f=1}^{N} \frac{\left(1+r^f+2g\sqrt{r^f}\right)}{r^{f\left(\frac{1}{2}+\varepsilon\right)[N/f]}\left(r^{\left(\frac{1}{2}+\varepsilon\right)f}-1\right)} \leqslant$$

$$\leqslant 8\sum_{f=1}^{[N/2]} \frac{r^{\left(\frac{1}{2}-\varepsilon\right)f}+2g\,r^{-f\varepsilon}}{r^{(N-f)\left(\frac{1}{2}+\varepsilon\right)}} + 8\sum_{f>[N/2]}^{N} \frac{r^{\left(\frac{1}{2}-\varepsilon\right)f}+2g\,r^{-f\varepsilon}}{r^{f\left(\frac{1}{2}+\varepsilon\right)}} \leqslant$$

$$\leqslant \frac{8}{r^{N\left(\frac{1}{2}+\varepsilon\right)}} \sum_{f=1}^{[N/2]} (r^f+2g\,r^{\frac{f}{2}}) + 8\sum_{f>[N/2]} (r^{-2\varepsilon f}+2g\,r^{-\left(\frac{1}{2}+2\varepsilon\right)f}) \leqslant$$

$$\leqslant \frac{8}{r^{N\left(\frac{1}{2}+\varepsilon\right)}}\left(\frac{r^{\frac{N}{2}+1}-r}{r-1}+2g\frac{r^{\frac{N}{4}+\frac{1}{2}}-r^{\frac{1}{2}}}{r^{\frac{1}{2}}-1}\right) + \frac{8r^{-\varepsilon N}}{1-r^{-2\varepsilon}} + \frac{16gr^{-\frac{N}{4}-\varepsilon N}}{1-r^{-\frac{1}{2}-2\varepsilon}} \leqslant$$

$$\leqslant \frac{64}{r^{\varepsilon N}}\left(2gr^{-\frac{N}{4}}+\frac{1}{r^{\varepsilon}-1}+1\right) \leqslant \frac{128}{r^{\varepsilon N}}\left(g\,r^{-\frac{N}{4}}+\frac{1+\varepsilon}{\varepsilon}\right).$$

Calculation of S_1:

$$S_1(N,\varepsilon) = r^{\frac{1}{2}-\varepsilon} \cdot \frac{r^{\left(\frac{1}{2}-\varepsilon\right)N}-1}{r^{\frac{1}{2}-\varepsilon}-1} = \frac{r^{\left(\frac{1}{2}-\varepsilon\right)N}-1}{1-r^{\varepsilon-\frac{1}{2}}}.$$

Calculation of S_2:

$$0 \leqslant |S_2(N,\varepsilon)| \leqslant \frac{1-r^{-\left(\frac{1}{2}+\varepsilon_0\right)N}}{r^{\frac{1}{2}+\varepsilon_0}-1} \leqslant 4.$$

Calculation of S_3:

$$R_3(N,\varepsilon) = S_3(N,\varepsilon) - \sum_{j=1}^{g}\left(\frac{\pi_j}{r^{\frac{1}{2}+\varepsilon}-\pi_j}+\frac{\bar{\pi}_j}{r^{\frac{1}{2}+\varepsilon}-\bar{\pi}_j}\right) =$$

$$= -\sum_{j=1}^{g}\sum_{n=N+1}^{\infty}\left(\frac{\pi_j}{r^{\frac{1}{2}+\varepsilon}}\right)^n + \left(\frac{\bar{\pi}_j}{r^{\frac{1}{2}+\varepsilon}}\right)^n.$$

The absolute value of the right hand side can be bounded using the fact that $|\pi_j| \leqslant r^{\frac{1}{2}}$:

$$|R_3(N,\varepsilon)| = \left|\sum_{j=1}^{g}\sum_{n=N+1}^{\infty}\left(\frac{\pi_j}{r^{\frac{1}{2}+\varepsilon}}\right)^n + \left(\frac{\bar{\pi}_j}{r^{\frac{1}{2}+\varepsilon}}\right)^n\right| \leqslant 2g\,\frac{r^{-N\varepsilon_0}}{r^{\varepsilon_0}-1} \leqslant 4g\,\frac{r^{-N\varepsilon_0}}{\varepsilon_0}.$$

From the expression (1.1) of $\zeta_K(s)$ as rational function in $t=r^{-s}$ we can easily deduce the following formula for its logarithmic derivative:

$$\frac{1}{\log r}Z_K\Big(\frac{1}{2}+\varepsilon\Big)=-\frac{1}{r^{\frac{1}{2}+\varepsilon}-1}-\frac{1}{r^{-\frac{1}{2}+\varepsilon}-1}+\sum_{j=1}^{g}\Big(\frac{\pi_j}{r^{\frac{1}{2}+\varepsilon}-\pi_j}+\frac{\bar{\pi}_j}{r^{\frac{1}{2}+\varepsilon}-\bar{\pi}_j}\Big).$$

Putting it all together we get the statement of the theorem. □

Remark 2.2. Using our theorem we can easily reprove the basic inequality from [15]. We take a real $\varepsilon<\frac{1}{4}$, and remark that

$$\frac{1}{\log r}Z_K\Big(\frac{1}{2}+\varepsilon\Big)+\frac{1}{r^{\frac{1}{2}+\varepsilon}-1}+\frac{1}{r^{-\frac{1}{2}+\varepsilon}-1}+g=$$
$$=\sum_{j=1}^{g}\Big(\frac{\pi_j}{r^{\frac{1}{2}+\varepsilon}-\pi_j}+\frac{\bar{\pi}_j}{r^{\frac{1}{2}+\varepsilon}-\bar{\pi}_j}+1\Big)\geqslant 0,$$

as

$$\frac{\pi_j}{r^{\frac{1}{2}+\varepsilon}-\pi_j}+\frac{\bar{\pi}_j}{r^{\frac{1}{2}+\varepsilon}-\bar{\pi}_j}+1=\frac{r^{1+2\varepsilon}-|\pi_j|^2}{(r^{\frac{1}{2}+\varepsilon}-\pi_j)(r^{\frac{1}{2}+\varepsilon}-\bar{\pi}_j)}\geqslant 0.$$

Now, from the theorem we get that

$$\sum_{f=1}^{N}\frac{f\Phi_{r^f}}{r^{\left(\frac{1}{2}+\varepsilon\right)f}-1}\leqslant g+O\Big(\frac{g}{\varepsilon r^{\varepsilon N}}\Big)+O(r^{\frac{N}{2}}).$$

We divide by g and first let $g\to\infty$ (varying K), after that we let $N\to\infty$ and finally we take the limit when $\varepsilon\to 0$. In doing so we obtain the basic inequality from [14]:

$$\sum_{f=1}^{\infty}\frac{f\phi_{r^f}}{r^{\frac{f}{2}}-1}\leqslant 1.$$

Remark 2.3. Using the explicit formulae due to Lachaud and Tsfasman, one can deal with the case of asymptotically good families of smooth projective absolutely irreducible algebraic varieties over finite fields.

3. Proof of Theorem 1.2

Our starting point will be the Weil explicit formula, the proof of which can be found in [10] or in [7, Ch. XVII] (with slightly more general conditions on the test functions).

Consider the class (W) of even real valued functions, satisfying the following conditions:

1) there exists $\varepsilon > 0$ such that $\int\limits_0^\infty F(x)e^{\left(\frac{1}{2}+\varepsilon\right)x}\,dx$ is convergent in the sense of Cauchy;

2) there exists $\varepsilon > 0$ such that $F(x)e^{\left(\frac{1}{2}+\varepsilon\right)x}$ has bounded variation;

3) $\dfrac{F(0)-F(x)}{x}$ has bounded variation;

4) for any x we have $F(x) = \dfrac{F(x-0)+F(x+0)}{2}$.

For such a function F we define

$$\phi(s) = \int\limits_{-\infty}^{+\infty} F(x)e^{(s-\frac{1}{2})x}\,dx. \tag{3.1}$$

The Weil explicit formula for Dedekind zeta functions of number fields can be stated as follows:

Theorem 3.1 (Weil). *Let K be a number field. Let F belong to the class (W) and let $\phi(s)$ be defined by* (3.1). *Then the sum $\sum\limits_{|\operatorname{Im}\rho|<T} \phi(\rho)$, where ρ runs through the non-trivial zeroes of the Dedekind zeta function of K, is convergent when $T \to \infty$ and the limit $\sum\limits_{\rho} \phi(\rho)$ is given by*:

$$\sum_{\rho} \phi(\rho) = F(0)\Big(2g - n(\gamma + \log 8\pi) - r_1\frac{\pi}{2}\Big) + 4\int\limits_0^\infty F(x)\operatorname{ch}\Big(\frac{x}{2}\Big) +$$

$$+ r_1\int\limits_0^\infty \frac{F(0)-F(x)}{2\operatorname{ch}(\frac{x}{2})}\,dx + n\int\limits_0^\infty \frac{F(0)-F(x)}{2\operatorname{sh}(\frac{x}{2})}\,dx - 2\sum_{\mathfrak{p},m} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{\frac{m}{2}}} F(m\log \mathrm{N}\mathfrak{p}), \tag{3.2}$$

where the last sum is taken over all prime ideals $\mathfrak{p}$ in K and all integers $m \geqslant 1$.

First of all, we remark that, if we have a complex valued function $F(x)$ with both real and imaginary parts $F_0(x)$ and $F_1(x)$ being even and lying in (W), we can apply (3.2) separately to $F_0(x)$ and $F_1(x)$. Thus the explicit formula, being linear in the test function, is also applicable to the initial complex valued function $F(x)$.

We apply the explicit formula to the function defined by

$$F_{N,\varepsilon}(x) = \begin{cases} e^{-\varepsilon|x|} & \text{if } |x| < \log(N+\frac{1}{2}), \\ 0 & \text{if } |x| > \log(N+\frac{1}{2}) \end{cases}$$

(here $N + \frac{1}{2}$ is take to avoid counting some of the terms with the factor $\frac{1}{2}$).

Next, we estimate each of the terms in (3.2).

3.1. The sum over the primes.

$$\sum_{\mathfrak{p},m} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{\frac{m}{2}}} F_{N,\varepsilon}(m \log \mathrm{N}\mathfrak{p}) = \sum_{\mathrm{N}\mathfrak{p}^m \leqslant N} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{\left(\frac{1}{2}+\varepsilon\right)m}} =$$
$$= \sum_{\mathrm{N}\mathfrak{p} \leqslant N} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{\frac{1}{2}+\varepsilon} - 1} - \sum_{\mathrm{N}\mathfrak{p} \leqslant N} \log \mathrm{N}\mathfrak{p} \sum_{m > \frac{\log N}{\log \mathrm{N}\mathfrak{p}}} \frac{1}{\mathrm{N}\mathfrak{p}^{\left(\frac{1}{2}+\varepsilon\right)m}}.$$

We have to estimate the sum:

$$\Delta(N,\varepsilon) = \sum_{\mathrm{N}\mathfrak{p} \leqslant N} \log \mathrm{N}\mathfrak{p} \sum_{m > \frac{\log N}{\log \mathrm{N}\mathfrak{p}}} \frac{1}{\mathrm{N}\mathfrak{p}^{\left(\frac{1}{2}+\varepsilon\right)m}}.$$

Taking the absolute values, we can assume that ε is real. Calculating the remainder term of the geometric series, we get:

$$\Delta(N,\varepsilon) \leqslant (2+\sqrt{2}) \sum_{\mathrm{N}\mathfrak{p} \leqslant N} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{\left(\frac{1}{2}+\varepsilon\right)\left(\left[\frac{\log N}{\log \mathrm{N}\mathfrak{p}}\right]+1\right)}}$$

(for $(1-\mathrm{N}\mathfrak{p}^{-1/2-\varepsilon})^{-1} \leqslant (1-2^{-1/2})^{-1} \leqslant \sqrt{2}(1+\sqrt{2})$).

Let us split the sum into two parts according as whether $\mathrm{N}\mathfrak{p} > \sqrt{N}$ or not. Taking into account that $\log \mathrm{N}\mathfrak{p}[\log N/\log \mathrm{N}\mathfrak{p}] \geqslant \log N - \log \mathrm{N}\mathfrak{p}$ for $\log \mathrm{N}\mathfrak{p} \leqslant [\log \sqrt{\mathrm{N}\mathfrak{p}}]$, we obtain:

$$\Delta(N,\varepsilon) \leqslant (2+\sqrt{2})\left(\sum_{\mathrm{N}\mathfrak{p} \leqslant \sqrt{N}} \frac{\log \mathrm{N}\mathfrak{p}}{e^{\log \mathrm{N}(\frac{1}{2}+\varepsilon)}} + \sum_{\sqrt{N} < \mathrm{N}\mathfrak{p} \leqslant N} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{(1+2\varepsilon)}}\right).$$

Write

$$\Delta_1(N,\varepsilon) = \sum_{\mathrm{N}\mathfrak{p} \leqslant \sqrt{N}} \frac{\log \mathrm{N}\mathfrak{p}}{e^{\log \mathrm{N}(\frac{1}{2}+\varepsilon)}}, \quad \Delta_2(N,\varepsilon) = \sum_{\sqrt{N} < \mathrm{N}\mathfrak{p} \leqslant N} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{(1+2\varepsilon)}}.$$

For $\Delta_1(N,\varepsilon)$ we have:

$$\Delta_1(N,\varepsilon) \leqslant \frac{1}{N^{\frac{1}{2}+\varepsilon}} \sum_{\mathrm{N}\mathfrak{p} \leqslant \sqrt{N}} \log \mathrm{N}\mathfrak{p}.$$

The last sum can be estimated with the help of Lagarias and Odlyzko results (which use GRH, cf. [6, Theorem 9.1]):

$$\sum_{\mathrm{N}\mathfrak{p} \leqslant \sqrt{N}} \log \mathrm{N}\mathfrak{p} \leqslant \sum_{\mathrm{N}\mathfrak{p}^k \leqslant \sqrt{N}} \log \mathrm{N}\mathfrak{p} = \sqrt{N} + O(N^{\frac{1}{4}} \log N(g + n\log N))$$

with an effectively computable absolute constant in O. Thus we get:

$$\Delta_1(N,\varepsilon) \leqslant \frac{2+\sqrt{2}}{N^{\varepsilon}} + a_0 \frac{g\log N + n\log^2 N}{N^{\frac{1}{4}+\varepsilon}}.$$

We can estimate the sum $\Delta_2(N,\varepsilon)$ as follows:

$$\Delta_2(N,\varepsilon) \leqslant \int\limits_{\sqrt{N}}^{\infty} \frac{\log t}{t^{1+2\varepsilon}}\, d\pi(t),$$

where $\pi(t)$ is the prime counting function $\pi(t) = \sum\limits_{\mathrm{Np}\leqslant t} 1$. As before, according to Lagarias and Odlyzko, $\pi(t) = \int\limits_{2}^{t} \frac{dx}{\log x} + \delta(t)$, with $|\delta(t)| \leqslant$ $\leqslant a_1\sqrt{t}(g + n\log t)$. Thus, substituting, we get:

$$\Delta_2(N,\varepsilon) \leqslant \int\limits_{\sqrt{N}}^{\infty} t^{-1-2\varepsilon}\, dt + 2|\delta(\sqrt{N})| \frac{\log N}{N^{\frac{1}{2}+\varepsilon}} + \left| \int\limits_{\sqrt{N}}^{\infty} \delta(t) \frac{1-(1+2\varepsilon)\log t}{t^{2+2\varepsilon}}\, dt \right|.$$

We deduce that

$$\Delta_2(N,\varepsilon) \leqslant \frac{1}{2\varepsilon N^{\varepsilon}} + 2a_1(g+n\log N)\frac{\log N}{N^{\frac{1}{4}+\varepsilon}} + \\ + \int\limits_{\sqrt{N}}^{\infty} a_1(g+n\log t)\frac{|1-(1+2\varepsilon)\log t|}{t^{\frac{3}{2}+2\varepsilon}}\, dt.$$

For $N \geqslant 8$ we have:

$$\int\limits_{\sqrt{N}}^{\infty} a_1(g+n\log t)\frac{|1-(1+2\varepsilon)\log t|}{t^{\frac{3}{2}+2\varepsilon}}\, dt \leqslant \int\limits_{\sqrt{N}}^{\infty} a_1(g+n\log t)\frac{(1+2\varepsilon)\log t}{t^{\frac{3}{2}+2\varepsilon}}\, dt.$$

Integrating by parts, we can find that

$$\int\limits_{\sqrt{N}}^{\infty} \frac{\log t}{t^{\frac{3}{2}+2\varepsilon}}\, dt = \frac{\log N}{2\left(\frac{1}{2}+2\varepsilon\right)N^{\frac{1}{4}+\varepsilon}} + \frac{1}{\left(\frac{1}{2}+2\varepsilon\right)^2 N^{\frac{1}{4}+\varepsilon}},$$

and

$$\int\limits_{\sqrt{N}}^{\infty} \frac{\log^2 t}{t^{\frac{3}{2}+2\varepsilon}}\, dt = \frac{\log^2 N}{4\left(\frac{1}{2}+2\varepsilon\right)N^{\frac{1}{4}+\varepsilon}} + \frac{\log N}{2\left(\frac{1}{2}+2\varepsilon\right)^2 N^{\frac{1}{4}+\varepsilon}} + \frac{1}{\left(\frac{1}{2}+2\varepsilon\right)^3 N^{\frac{1}{4}+\varepsilon}}.$$

We conclude that the following estimate holds:

$$\Delta_2(N,\varepsilon) \leqslant \frac{1}{2\varepsilon N^{\varepsilon}} + a_2\left(\frac{n\log^2 N}{N^{\frac{1}{4}+\varepsilon}} + \frac{g\log N}{N^{\frac{1}{4}+\varepsilon}}\right).$$

Putting everything together, we see that:

$$|\Delta(N,\varepsilon)| \ll \frac{1}{\varepsilon_0 N^{\varepsilon_0}} + \frac{\log N}{N^{\frac{1}{4}+\varepsilon_0}}(n\log N + g). \tag{3.3}$$

3.2. Archimedean terms

First of all,

$$2\int_0^\infty F_{N,\varepsilon}(x)\operatorname{ch}\Big(\frac{x}{2}\Big)\,dx = 2\int_0^{\log(N+\frac{1}{2})} e^{-\varepsilon x}\operatorname{ch}\Big(\frac{x}{2}\Big)\,dx =$$

$$= \frac{\Big(N+\dfrac{1}{2}\Big)^{\frac{1}{2}-\varepsilon}-1}{\dfrac{1}{2}-\varepsilon}+O(1). \tag{3.4}$$

Let

$$I_{N,\varepsilon} = \int_0^\infty \frac{1-F_{N,\varepsilon}(x)}{2\operatorname{sh}(\frac{x}{2})}\,dx \quad \text{and} \quad I_{\infty,\varepsilon} = \int_0^\infty \frac{1-e^{-\varepsilon x}}{2\operatorname{sh}(\frac{x}{2})}\,dx.$$

We have for $N \geqslant 4$:

$$|I_{\infty,\varepsilon} - I_{N,\varepsilon}| \leqslant \int_{\log N}^\infty \frac{2}{e^{\frac{x}{2}}}\,dx \leqslant \frac{4}{\sqrt{N}}.$$

Now,

$$I_{\infty,\varepsilon} = \int_0^\infty \Big(\frac{e^{-\frac{x}{2}}}{1-e^{-x}} - \frac{e^{-(\frac{1}{2}+\varepsilon)x}}{1-e^{-x}}\Big)\,dx =$$

$$= \int_0^\infty \Big(\Big(\frac{e^{-\frac{x}{2}}}{1-e^{-x}} - \frac{e^{-x}}{x}\Big) + \Big(\frac{e^{-x}}{x} - \frac{e^{-(\frac{1}{2}+\varepsilon)x}}{1-e^{-x}}\Big)\Big)\,dx = \psi\Big(\frac{1}{2}+\varepsilon\Big) - \psi\Big(\frac{1}{2}\Big),$$

as

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = \int_0^\infty \Big(\frac{e^{-t}}{x} - \frac{e^{-xt}}{1-e^{-t}}\Big)\,dt.$$

The second integral

$$J_{N,\varepsilon} = \int_0^\infty \frac{1-F_{N,\varepsilon}(x)}{2\operatorname{ch}(\frac{x}{2})}\,dx$$

can be estimated along the same lines using an integral from [1, 3.541]:

$$\int_0^\infty \frac{e^{-\varepsilon x}}{\operatorname{ch}(\frac{x}{2})}\,dx = \psi\Big(\frac{1}{4}+\frac{\varepsilon}{2}\Big) - \psi\Big(\frac{3}{4}+\frac{\varepsilon}{2}\Big).$$

Taking into account that $\psi(2x)=\frac{1}{2}\Big(\psi(x)+\psi\Big(x+\frac{1}{2}\Big)\Big)+\log 2$, we finally obtain:

$$\begin{aligned} J_{N,\varepsilon} &= \frac{\pi}{2}+\log 2+\psi\Big(\frac{1}{4}+\frac{\varepsilon}{2}\Big)-\psi\Big(\frac{1}{2}+\varepsilon\Big)+O\Big(\frac{1}{\sqrt{N}}\Big), \\ I_{N,\varepsilon} &= \gamma+\log 4+\psi\Big(\frac{1}{2}+\varepsilon\Big)+O\Big(\frac{1}{\sqrt{N}}\Big). \end{aligned} \tag{3.5}$$

3.3. The sum over the zeroes: the main term

Let us estimate now the sum $\sum_\rho \phi(\rho)$ over zeroes of $\zeta_K(s)$. Let $\rho = = \frac{1}{2}+it$ be a zero of the zeta function of K on the critical line. Put $y=\log\Big(N+\frac{1}{2}\Big)$. We have

$$\phi(\rho) = \int_{-y}^{y} e^{-\varepsilon|x|+itx}\,dx = \int_0^y e^{(-\varepsilon+it)x}\,dx+\int_0^y e^{(-\varepsilon-it)x}\,dx,$$

so

$$\phi(\rho) = \frac{2}{\varepsilon^2+t^2}(\varepsilon+e^{-\varepsilon y}(-\varepsilon\cos(ty)+t\sin(ty))).$$

We divide the sum over ρ into three parts:

$$\begin{aligned} S_1(\varepsilon) &= \sum_{\rho=\frac{1}{2}+it}\frac{\varepsilon}{\varepsilon^2+t^2}; \\ S_2(y,\varepsilon) &= \sum_{\rho=\frac{1}{2}+it}\frac{\cos(ty)}{\varepsilon^2+t^2}; \\ S_3(y,\varepsilon) &= \sum_{\rho=\frac{1}{2}+it}\frac{t\sin(ty)}{\varepsilon^2+t^2}; \end{aligned}$$

so that

$$\sum_\rho \phi(\rho) = 2S_1(\varepsilon)-2\varepsilon e^{-\varepsilon y}S_2(y,\varepsilon)+2e^{-\varepsilon y}S_3(y,\varepsilon).$$

Let us relate the sum $S_1(\varepsilon)$ to $Z_K(s)$, the logarithmic derivative of $\zeta_K(s)$. Stark's formula (cf. [13, (9)]) gives us the following:

$$\begin{aligned} &\sum_\rho \frac{1}{s-\rho} = \\ &= \frac{1}{s-1}+\frac{1}{s}+g-\frac{n}{2}\log\pi+\frac{r_1}{2}\psi\Big(\frac{s}{2}\Big)+r_2(\psi(s)-\log 2)+Z_K(s), \end{aligned} \tag{3.6}$$

where as before $\psi(s)=\dfrac{\Gamma'(s)}{\Gamma(s)}$. Specializing at $s=\dfrac{1}{2}+\varepsilon$, we obtain:

$$\sum_{\rho=\frac{1}{2}+it}\frac{\varepsilon}{\varepsilon^2+t^2}=\frac{1}{\varepsilon-\frac{1}{2}}+\frac{1}{\varepsilon+\frac{1}{2}}+g-\frac{n}{2}\log\pi-r_2\log 2+$$
$$+\frac{r_1}{2}\psi\Big(\frac{1}{4}+\frac{\varepsilon}{2}\Big)+r_2\psi\Big(\frac{1}{2}+\varepsilon\Big)+Z_K\Big(\frac{1}{2}+\varepsilon\Big). \quad (3.7)$$

We note that the archimedean factors from the Stark formula and from the initial Weil explicit formula cancel each other. We are left to prove that $S_2(y,\varepsilon)$ and $S_3(y,\varepsilon)$ are sufficiently small.

3.4. The sum over the zeroes: the remainder term.

To estimate

$$S_2(y,\varepsilon)=\sum_{\rho=\frac{1}{2}+it}\frac{\cos(ty)}{\varepsilon^2+t^2}$$

we take the absolute values of all the terms in the sum so that

$$|S_2(y,\varepsilon)|\leqslant\sum_{\rho=\frac{1}{2}+it}\frac{1}{|\varepsilon^2+t^2|}\leqslant\sum_{\rho=\frac{1}{2}+it}\frac{n(j)}{\varepsilon_0^2+(t-|\varepsilon_1|)^2}, \quad (3.8)$$

where $n(j)$ is the number of zeroes with $|t-j|<1$. A standard estimate from [6, Lemma 5.4] yields $n(j)\ll g+n\log(j+2)$, thus

$$|S_2(y,\varepsilon)|\ll$$
$$\ll\frac{g+n\log(|\varepsilon_1|+2)}{\varepsilon_0^2}+g+n\sum_{j=1}^{|\varepsilon_1|+1}\frac{\log j}{|\varepsilon_1|+2-j}+g+n\log(|\varepsilon_1|+2)\ll$$
$$\ll(g+n\log^2(|\varepsilon_1|+2))\Big(1+\frac{1}{\varepsilon_0^2}\Big).$$

Let us finally estimate

$$S_3(y,\varepsilon)=\sum_{\rho=\frac{1}{2}+it}\frac{t\sin(ty)}{\varepsilon^2+t^2}.$$

We have

$$S_3(y,\varepsilon)=\sum_{\rho=\frac{1}{2}+it}\frac{\sin ty}{t}-\sum_{\rho=\frac{1}{2}+it}\frac{\varepsilon^2\sin(ty)}{t(\varepsilon^2+t^2)}=A(y)-B(y,\varepsilon).$$

The series for the formal derivative of $B(y,\varepsilon)$ with respect to y is given by

$$\sum_{\rho=\frac{1}{2}+it}\frac{\varepsilon^2\cos(ty)}{\varepsilon^2+t^2}.$$

Using the estimates for $S_2(y,\varepsilon)$ we deduce that on any compact subset of $[0,+\infty)$ this series is absolutely and uniformly convergent to $B'(y)$, and we have $|B'(y,\varepsilon)| \ll |\varepsilon|^2(g+n\log^2(|\varepsilon_1|+2))\Big(1+\frac{1}{\varepsilon_0^2}\Big)$. Thus we see that $|B(y)| \ll y|\varepsilon|^2(g+n\log^2(|\varepsilon_1|+2))\Big(1+\frac{1}{\varepsilon_0^2}\Big)$, since $B(0,\varepsilon)=0$.

3.5. The sum over the zeroes: the difficult part

We are left to estimate the term $A(y)$.

Let us recall a particular case of Weil explicit formula which is due to Landau (cf. [8]):

$$\sum_\rho \frac{x^\rho}{\rho} = x - \Psi(x) - r\log x - b - \frac{r_1}{2}\log(1-x^{-2}) - r_2\log(1-x^{-1}), \quad (3.9)$$

where $\Psi(x) = \sum_{N\mathfrak{p}^k \leqslant x} \log N\mathfrak{p}$, b is the constant term of the expansion of $Z_K(s)$ at 0, $r = r_1 + r_2 - 1$ and x is not a prime power. This formula is stated in [8] for $x \geqslant \frac{3}{2}$, however, applying Theorem 3.1 to the function

$$F_x(y) = \begin{cases} e^{|y|/2} & \text{if } |y| < \log x, \\ 0 & \text{if } |y| > \log x, \end{cases}$$

one can see that it is valid for any $x > 1$. We also note that by an effective version of the prime ideals theorem ([6, Theorem 9.1]) we have the following estimate:

$$\Psi(x) - x = O\Big(x^{\frac{1}{2}}\log x(g+n\log x)\Big). \quad (3.10)$$

Now, we introduce

$$C(x) = \sum_\rho \frac{x^\rho}{\rho}, \quad D(x) = \sum_{\rho\neq\frac{1}{2}} \frac{x^\rho}{\rho-\frac{1}{2}} \quad \text{and} \quad E(x) = D(x) - C(x).$$

From (3.9) and (3.10) we see that $C(x)$ is an integrable function on compact subsets of $(1,+\infty)$. Using the arguments similar to those from the previous subsection we can deduce that the series for $E(x)$ is absolutely and uniformly convergent on compact subsets of $[1,+\infty)$ and thus $E(x)$ is a continuous function on this interval. From this we conclude that the series for $D(x)$ is also convergent to a locally integrable function.

If we put $x = e^y$, we get

$$\operatorname{Re} D(e^y) = e^{\frac{y}{2}} \sum_{\rho\neq\frac{1}{2}} \frac{\sin(ty)}{t},$$

which is equal to $e^{\frac{y}{2}}A(y)$ up to a term corresponding to a possible zero of $\zeta_K(s)$ at $\rho=\frac{1}{2}$.

Since the series for $C(x)$ is not uniformly convergent, we will have to work with distributions defined by $C(x), D(x)$ and $E(x)$. See [11] for the basic notions and results used here. From the fact that a convergent series of distributions can be differentiated term by term we deduce that the following equality holds:

$$\frac{d}{dx}\frac{E(x)}{\sqrt{x}} = \frac{C(x)}{2\sqrt{x^3}}.$$

We apply (3.9) to the right hand side of this formula and integrate from $1+\delta$ to x (here $\delta>0$). The obtained equality will be valid in the sense of distributions, thus almost everywhere for the corresponding locally integrable functions defining these distributions. Since $E(x)$ is continuous, we see that the resulting identity

$$\frac{E(x)}{\sqrt{x}} = E(1+\delta)+\int\limits_{1+\delta}^{x}\frac{t-\Psi(t)}{2t^{\frac{3}{2}}}\,dt - r\int\limits_{1+\delta}^{x}\frac{\log t}{2t^{\frac{3}{2}}}\,dt -$$
$$-\int\limits_{1+\delta}^{x}\frac{b}{2t^{\frac{3}{2}}}\,dt-\frac{r_1}{2}\int\limits_{1+\delta}^{x}\frac{\log(1-t^{-2})}{2t^{\frac{3}{2}}}\,dt-r_2\int\limits_{1+\delta}^{x}\frac{\log(1-t^{-1})}{2t^{\frac{3}{2}}}\,dt$$

actually holds pointwise on $[1+\delta,+\infty)$. We use (3.10) to estimate $t-\Psi(t)$. It is easily seen that all the integrals converge when $\delta\to 0$. From [8, 10.RH] it follows that $b\ll g+n$.

$$E(1) = \sum_{\rho\neq\frac{1}{2}}\frac{1}{\rho-\frac{1}{2}}-\sum_{\rho}\frac{1}{\rho} = -\frac{1}{2}\sum_{\rho=\frac{1}{2}+it}\frac{1}{\frac{1}{4}+t^2},$$

the first sum being zero as the term in ρ and $1-\rho$ cancel each other. An estimate for the last sum can be made using (3.8). This gives $|E(1)|\ll g+n$. Putting it all together we see that $|E(x)|\ll\sqrt{x}\log^2 x(g+n\log x)$. The estimate $|C(x)|\ll\sqrt{x}\log^2 x(n+g)$ can be obtained directly using (3.10). Thus, we conclude that $|A(y)|\ll y^2(g+ny)$.

Finally, combining all together we get:

$$\sum_{\rho}\phi(\rho) = 2S_1(\varepsilon)+O\Big(\frac{|\varepsilon|^4+|\varepsilon|}{\varepsilon_0^2}(g+n\log N)\frac{\log^2 N}{N^{\varepsilon_0}}\Big).$$

This estimate together with (3.3), (3.4), (3.5) and (3.7) completes the proof of the theorem. □

Remark 3.2. Using our theorem we can derive the basic inequality from [16]. Indeed, we apply the formula (3.7) to express $Z_K\left(\frac{1}{2}+\varepsilon\right)$ via the series $\sum\limits_{\rho=\frac{1}{2}+it} \frac{\varepsilon}{\varepsilon^2+t^2}$ plus some archimedean terms. For a real positive $\varepsilon<\frac{1}{4}$ the latter sum is non-negative, thus we see that

$$\sum_{q\leqslant N} \frac{\Phi_q \log q}{q^{\frac{1}{2}+\varepsilon}-1} + \frac{n}{2}\log\pi + r_2\log 2 - \frac{r_1}{2}\psi\left(\frac{1}{4}+\frac{\varepsilon}{2}\right) - r_2\psi\left(\frac{1}{2}+\varepsilon\right) \leqslant$$
$$\leqslant g + O\left((g+n\log N)\frac{\log^2 N}{\varepsilon N^{\varepsilon}}\right) + O\left(\sqrt{N}\right).$$

Now, we divide by g and first let $g\to\infty$ (varying K), after that we let $N\to\infty$ and finally we take the limit when $\varepsilon\to 0$. Taking into account that

$$\psi\left(\frac{1}{2}\right) = -\gamma - 2\log 2 \quad\text{and}\quad \psi\left(\frac{1}{4}\right) = -\frac{\pi}{2} - \gamma - 3\log 2,$$

we obtain the basic inequality from [14]:

$$\sum_{q} \frac{\phi_q \log q}{\sqrt{q}-1} + \phi_{\mathbb{R}}\left(\log(2\sqrt{2\pi}) + \frac{\pi}{4} + \frac{\gamma}{2}\right) + \phi_{\mathbb{C}}(\log(8\pi)+\gamma) \leqslant 1.$$

Remark 3.3. The choice of the test functions $F_{N,\varepsilon}(x)$ in the explicit formula is not accidental. Indeed, the resulting formulas "approximate" the Stark formula (3.6) when $N\to\infty$.

4. Proof of Theorem 1.4 and of the corollaries

We will carry out the proofs in the function field case, the calculations in the number field case being exactly the same.

Proof of the Corollary 1.3. Assume first that $\varepsilon \neq \frac{1}{2} + \frac{2\pi i k}{\log r}$, $k\in\mathbb{Z}$. We note that

$$\left|\sum_{f=1}^{\infty} \frac{f\phi_{r^f}}{r^{\left(\frac{1}{2}+\varepsilon\right)f}-1} + \frac{1}{g_j \log r} Z_{K_j}\left(\frac{1}{2}+\varepsilon\right)\right| \leqslant$$
$$\leqslant \left|\sum_{f=N+1}^{\infty} \frac{f\phi_{r^f}}{r^{\left(\frac{1}{2}+\varepsilon\right)f}-1}\right| + \sum_{f=1}^{N} \frac{f\left|\frac{\Phi_{r^f}}{g_j} - \phi_{r^f}\right|}{r^{\left(\frac{1}{2}+\varepsilon\right)f}-1} +$$
$$+ \frac{1}{g_j}\left|\sum_{f=1}^{N} \frac{f\Phi_{r^f}}{r^{\left(\frac{1}{2}+\varepsilon\right)f}-1} + \frac{1}{\log r} Z_{K_j}\left(\frac{1}{2}+\varepsilon\right)\right|.$$

Given $\delta > 0$ we choose an integer N such that the first sum is less than δ (this is possible due to the basic inequality) and such that $\frac{1}{r^{\varepsilon_0 N}}\left(1+\frac{1}{\varepsilon_0}\right) \leqslant \delta$. Now, taking g sufficiently large, and using Theorem 1.1 as well as the convergence of $\frac{\Phi_{r^f}}{g_j}$ to ϕ_{r^f}, we conclude that the whole sum is $\ll \delta$. Thus, we deduce that

$$\lim_{j\to\infty} \frac{Z_{K_j}\left(\frac{1}{2}+\varepsilon\right)}{g_j} = Z_{\{K_j\}}\left(\frac{1}{2}+\varepsilon\right). \tag{4.1}$$

Now, the corollary immediately follows from Theorem 1.1 and (4.1). Though we initially assumed that $\varepsilon \neq \frac{1}{2} + \frac{2\pi i k}{\log r}$, the statement still holds for $\varepsilon = \frac{1}{2} + \frac{2\pi i k}{\log r}$ as all the function are continuous (and even analytic) for $\operatorname{Re}\varepsilon > 0$. □

Remark 4.1. The formula (4.1) no longer holds when $\varepsilon = 0$ as can be seen from the fact that $Z_K\left(\frac{1}{2}\right) = g_K - 1$. In fact, the identity holds if and only if our family is asymptotically optimal. Whether it holds or not for the logarithm of $\zeta_K(s)$ and not for its derivative seems to be very difficult to say at the moment. Even for quadratic fields this question is far from being obvious. It is known that in the number field case there exists a sequence (d_i) in $\mathbb{N}$ of density at least $\frac{1}{2}$ such that

$$\lim_{i\to\infty} \frac{\log \zeta_{\mathbb{Q}(\sqrt{d_i})}\left(\frac{1}{2}\right)}{\log d_i} = 0$$

(cf. [3]). The techniques of the evaluation of mollified moments of Dirichlet L- functions used in that paper is rather involved. In general one can prove an upper bound for the limit (cf. [17]). This is analogous to the "easy" inequality in the classical Brauer—Siegel theorem.

The interest of the question about the behaviour of $\log Z_K\left(\frac{1}{2}\right)$ can be in particular explained by its connection to the behaviour of the order of the Shafarevich—Tate group and the regulator of constant supersingular elliptic curves over function fields, the connection being provided by the Birch and Swinnerton-Dyer conjecture. In general, a similar question can be asked about the behaviour of these invariants in arbitrary families of elliptic curves. Some discussion on the problem is given in [4] (beware, however, that the proof of the main result there cannot be seen as a correct one as the change of limits, which is a key point, is not justified).

Proof of Theorem 1.4. It follows from the basic inequality that the series defining $\log\zeta_{\{K_i\}}(s)$ converges absolutely for $\operatorname{Re} s \geqslant \frac{1}{2}$. The function $\log\zeta_{\{K_i\}}(s)$ has a Dirichlet series expansion with positive coefficients, converging for $\operatorname{Re} s \geqslant \frac{1}{2}$. Thus, from a standard theorem on Dirichlet series (cf. [2, Lemma 5.56]), it must converge in some open domain $\operatorname{Re} s > \frac{1}{2} - \delta_0$ for $\delta_0 > 0$, defining an analytic function there. It follows that in the same domain the series for $Z_{\{K_i\}}(s)$ converges. Taking any δ with $0 < \delta < \delta_0$ we obtain:

$$\Bigl|\sum_{f=1}^{N} \frac{f\phi_{r^f}}{r^{\frac{f}{2}}-1} - \frac{1}{\log r} Z_{\{K_i\}}\Bigl(\frac{1}{2}\Bigr)\Bigr| = \Bigl|\sum_{f=N+1}^{\infty} \frac{f\phi_{r^f}}{r^{(\frac{1}{2}-\delta)f}-1} \cdot \frac{r^{(\frac{1}{2}-\delta)f}-1}{r^{\frac{f}{2}}-1}\Bigr| \leqslant$$
$$\leqslant \Bigl|\sum_{f=1}^{\infty} \frac{f\phi_{r^f}}{r^{(\frac{1}{2}-\delta)f}-1}\Bigr| \cdot \frac{r^{(\frac{1}{2}-\delta)N}-1}{r^{\frac{N}{2}}-1} O(r^{-\delta N}).$$

This gives the necessary result. □

Proof of the Corollary 1.5. We use Theorem 1.4 to obtain the necessary estimate much in the same spirit as in the proof of Theorem 1.4 itself. Using the function field Brauer—Siegel theorem to find the value for κ, we get:

$$\Bigl|\sum_{f=1}^{N} \phi_{r^f} \log\frac{r^f}{r^f-1} - \kappa\Bigr| = \Bigl|\sum_{f=N+1}^{\infty} \frac{f\phi_{r^f}}{r^{\frac{f}{2}}-1} \cdot \frac{r^{\frac{f}{2}}-1}{f} \cdot \log\frac{r^f}{r^f-1}\Bigr| \leqslant$$
$$\leqslant \Bigl|\sum_{f=N+1}^{\infty} \frac{f\phi_{r^f}}{r^{\frac{f}{2}}-1}\Bigr| \cdot \frac{r^{\frac{N}{2}}-1}{N} \cdot \log\frac{r^N}{r^N-1} = O(r^{-\delta N}) \cdot O\Bigl(\frac{r^{-\frac{N}{2}}}{N}\Bigr).$$

Indeed, $N \mapsto \frac{1}{N}(r^{\frac{N}{2}}-1)\log\frac{r^N}{r^N-1}$ is decreasing for $N \geqslant 2$. The required estimate follows. □

Remark 4.2. Actually, our method gives an easy and conceptual proof of the explicit version of the Brauer—Siegel theorem from [9] (which is roughly speaking the statement of Corollary 1.5 with $\delta = 0$). It shows that the rate of convergence in the Brauer—Siegel theorem essentially depends on how far to the left the limit zeta function $\zeta_{\{K_i\}}(s)$ is analytic. In the number field case we even save $\log^2 N$ in the estimate of the error term compared to what is proven in [9].

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Asymptotic methods in number theory and algebraic geometry

(with P. Lebacque)

Abstract. The paper is a survey of recent developments in the asymptotic theory of global fields and varieties over them. First, we give a detailed motivated introduction to the asymptotic theory of global fields which is already well shaped as a subject. Second, we treat in a more sketchy way the higher dimensional theory where much less is known and many new research directions are available.

Résumé. Cet article est un survol des développements récents dans la théorie asymptotique des corps globaux et des variétés algébriques définies sur les corps globaux. Dans un premier temps, nous donnons une introduction détaillée et motivée à la théorie asymptotique des corps globaux, théorie déjà bien établie. Puis nous aborderons plus rapidement la théorie asymptotique en dimension supérieure où peu de choses sont connues et où bien des directions de recherche sont ouvertes.

1. Introduction: the origin of the asymptotic theory of global fields

The goal of this article is to give a survey of asymptotic methods in number theory and algebraic geometry developed in the last decades. The problems that are treated by the asymptotic theory of global fields (that is number fields or function fields) and varieties over them are quite diverse in nature. However, they are connected by the use of zeta functions, which play the key role in the asymptotic theory.

We begin by a very well known problem which lies at the origin of the asymptotic theory of global fields. Let $\mathbb{F}_r$ be the finite field with r elements. For a smooth projective curve C over $\mathbb{F}_r$ we let $N_r(C)$ be the number of $\mathbb{F}_r$-point on C. We denote by $g(C)$ be the genus of C. The problem consists of finding the maximum $N_r(g)$ of the numbers $N_r(C)$ over all smooth projective curves of genus g over $\mathbb{F}_r$: $N_r(g) = \max_{g(C)=g} N_r(C)$.

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The first upper bound was discovered by André Weil in 1940s as a direct consequence of his proof of the Riemann hypothesis for curves over finite fields. He showed that $N_r(C)$ satisfies the inequality

$$N_r(C) \leqslant r+1+2g\sqrt{r}.$$

Weil bound though extremely useful in many applications is far from being optimal. A dramatic search for the improvements of this bound and for the examples giving lower bounds on $N_r(g)$ has begun in 1980s with the discovery of Goppa that curves over finite fields with many points can be used to construct good error-correcting codes. To show how important the developments in this area were it suffices to mention the names of some mathematicians who turned their attention to these questions: J.-P. Serre, V. Drinfeld, Y. Ihara, H. Stark, R. Schoof, M. Tsfasman, S. Vlăduţ, G. van der Geer, K. Lauter, H. Stichtenoth, A. Garcia, etc.

As suggested in [33] by J.-P. Serre the cases when g is small and that when g is large require completely different treatment. That is the latter case which interests us in this article. The first major result in this direction was the following theorem of V. Drinfeld and S. Vlăduţ [5]:

Theorem 1.1 (Drinfeld—Vlăduţ). *For any family of smooth projective curves $\{C_i\}$ over $\mathbb{F}_r$ of growing genus we have* $\limsup_{i\to\infty} \dfrac{N_r(C_i)}{g(C_i)} \leqslant \sqrt{r}-1.$

Moreover, in the case, when r is a square this bound turns out to be optimal. The families of curves, attaining this bound are constructed in many different ways: modular curves, Drinfeld modular curves, explicit iterated constructions, etc. We refer the reader to Section 4 for more details. This result, significantly improved and then reinterpreted in terms of limit zeta functions by M. Tsfasman and S. Vlăduţ, lies at the very base of the asymptotic theory of global fields. We will discuss all this in detail in Section 2. It is also possible to extend the Drinfeld—Vlăduţ inequalities to the case of higher dimensional varieties. This serves as a keystone in the construction of the higher dimensional asymptotic theory (see Section 5).

We will now turn our attention to yet another source of development of the asymptotic theory, this time in the case of number fields. Let K be an algebraic number field, that is a finite extension of $\mathbb{Q}$. We denote by $n_K=[K:\mathbb{Q}]$ its degree, and by D_K its discriminant. An important question (both on its own account and due to its applications in various domains of number theory, arithmetic geometry and theory of sphere packings) is to know the rate of grows of discriminants of number fields. The first bound on D_K was obtained by H. Minkowsky using the geometry of numbers. This bound was improved more than half a century later by H. Stark, J.-P. Serre and A. Odlyzko ([35], [32], [29], [30]) who used analytic methods involving zeta functions. The bounds they prove are as follows:

Theorem 1.2 (Odlyzko). *For a family of number fields $\{K_i\}$ we have*

$$\log|D_{K_i}| \geqslant A\cdot r_1(K_i)+2B\cdot r_2(K_i)+o(n_{K_i}),$$

where $r_1(K_i)$ and $r_2(K_i)$ are respectively the number of real and complex places of K_i. Unconditionally, we can take $A=\log(4\pi)+\gamma+1\approx 60.8$, $B=\log(4\pi)+\gamma\approx 22.3$, and, assuming the generalized Riemann Hypothesis (GRH), one can take, $A=\log(8\pi)+\gamma+\frac{\pi}{2}\approx 215.3$, $B=\log(8\pi)+\gamma\approx$ ≈ 44.7, where $\gamma=0.577$ is Euler's gamma constant.

The fact that GRH drastically improves the results is omnipresent in the asymptotic theory of global fields. Fortunately, GRH is known for zeta functions of curves over finite fields (Weil bounds) and, more generally, of varieties over finite fields (Deligne's theorem), which allows to have both stronger results and simpler proofs in the case of positive characteristic.

M. Tsfasman and S. Vlăduţ managed to generalize the above inequalities taking into account the contribution of finite places of the fields. In fact, the restriction of the so-called basic inequality proven by M. Tsfasman and S. Vlăduţ to infinite primes gives us the inequalities of Odlyzko—Serre. If we restrict the basic inequality to finite places we obtain an analogue of the generalized Drinfeld—Vlăduţ inequality in the case of number fields. The reader will find more information on this in the next section of the paper.

The last, but not least, problem that led to the development of the asymptotic theory of global fields and varieties over them was the Brauer—Siegel theorem. Let h_K denote the class number of a number field K and let R_K be its regulator. The classical Brauer—Siegel theorem, proven by Siegel ([34]) in the case of quadratic fields and by Brauer ([3]) in general describes the behaviour of the product $h_K R_K$ in families of number fields. The initial motivation for it was a conjecture of Gauss on imaginary quadratic fields, however it has got many important applications elsewhere. The theorem can be stated as follows:

Theorem 1.3 (Brauer—Siegel). *For a family of number fields $\{K_i\}$ we have $\lim\limits_{i\to\infty}\dfrac{\log(h_{K_i}R_{K_i})}{\log\sqrt{|D_{K_i}|}}=1$ provided the family satisfies two conditions:*

(i) $\lim\limits_{i\to\infty}\dfrac{n_{K_i}}{g_{K_i}}=0$;

(ii) *either GRH holds, or all the fields K_i are normal over $\mathbb{Q}$.*

It is possible to remove the first and relax the second conditions of the theorem. The first step towards it was made by Y. Ihara in [13] who considered families of unramified number fields. A complete answer (at least modulo GRH) was given by M. Tsfasman and S. Vlăduţ in [40] who showed how to treat this problem in the framework of the asymptotic theory of number fields, in particular using the concept of limit zeta functions. The corresponding question for curves over finite fields is also of great interest since it describes the asymptotic behaviour of the number of rational points on Jacobians of curves over finite fields. All this will be discussed in detail in the Section 3.

In our introduction we mostly considered the one dimensional case of number fields or function fields. Here the theory is best developed. However, there is quite a number of results and conjectures for higher dimensional varieties with particularly nice arithmetical applications. Some of the results in this actively developing area are discussed in Section 5.

Let us finally say that, despite of the fact that the theory of error correcting codes and the theory of sphere packings are just briefly mentioned in our introduction their role in the creation of the asymptotic theory of global fields is fundamental. Indeed many questions some of which were mentioned here (maximal number of points on curves, growth of the discriminants, etc.) received particular attention due to their relation to error-correcting codes or sphere packings.

2. Basic concepts and results. Tsfasman—Vlăduţ invariants of infinite global fields

Many authors considered the behaviour of arithmetic data (decomposition of primes, genus, root discriminant, class number, regulator etc.) in families of global fields. Tsfasman and Vlăduţ laid the foundation for the asymptotic theory of global fields in order not to consider fields in a family, but the limit object (say, a limit zeta function) that would encode the information concerning the asymptotics of the initial arithmetic data.

In this section we introduce some definitions and give basic properties of families of global fields.

2.1. Tsfasman—Vlăduţ invariants

Arguments and proofs for the results from this subsection can be found in [40]. Let us first define the objects we are to work with. Let r be a power of a prime p, and let $\overline{\mathbb{F}}_r$ denote the algebraic closure of $\mathbb{F}_r$.

Definition 2.1. A family of global fields is a sequence $\mathcal{K}=\{K_n\}_{n\in\mathbb{N}}$ such that:

1) Either all the K_n are finite extensions of $\mathbb{Q}$ or all the K_n are finite extensions of $\mathbb{F}_r(t)$ with $\overline{\mathbb{F}}_r\cap K_n=\mathbb{F}_r$.

2) if $i\neq j$, K_i is not isomorphic to K_j.

A tower of global fields is a family satisfying in addition $K_n\subset K_{n+1}$ for every $n\in\mathbb{N}$. An infinite global (resp. number, resp. function) field is the limit of a tower of global (resp. number, resp. function) fields, i. e. it is the union $\bigcup\limits_{n=1}^{\infty} K_n$.

Definition 2.2. The genus g_K of a function field is the genus of the corresponding smooth projective curve. We define the genus of a number field K as $g_K=\log\sqrt{|D_K|}$, where D_K is the discriminant of K.

As there are (up to an isomorphism) only finitely many global fields with genus smaller than a fixed real number g, we have the following proposition.

Proposition 2.3. *For any family $\{K_i\}$ of global fields the genus $g_{K_i}\to\to+\infty$.*

Thus, in the number fields case, any infinite algebraic extension of $\mathbb{Q}$ is an infinite number field, whereas in the function fields case, we require the infinite algebraic extension of $\mathbb{F}_r(t)$ to contain a sequence of function fields with genus going to infinity.

Let us now define the so-called Tsfasman—Vlăduţ invariants of a family of global fields. Throughout the paper, we use the acronyms NF and FF for the number field and the function field cases respectively. As before, the GRH indication means that we assume the generalized Riemann Hypothesis for Dedekind zeta functions.

First we introduce some notation to be used throughout the paper:

$\mathcal{Q}$	the field $\mathbb{Q}$ (NF), $\mathbb{F}_r(t)$ (FF);
n_K	$[K:\mathbb{Q}]$;
D_K	discriminant of K (NF);
g_K	the genus of K (FF), the genus of K equal to $\log\sqrt{|D_K|}$ (NF);
$\mathrm{Pl}_f(K)$	the set of finite places of K;
$\mathrm{N}\mathfrak{p}$	the norm of a place $\mathfrak{p}\in\mathrm{Pl}_f(K)$;
$\deg\mathfrak{p}$	$\log_r \mathrm{N}\mathfrak{p}$ (FF);
$\Phi_q(K)$	the number of places of K of norm q;
$\Phi_{\mathbb{R}}(K)$	the number of real places of K (NF);
$\Phi_{\mathbb{C}}(K)$	the number of complex places of K (NF).

We consider the set of possible indices for the Φ_q,

$$A = \begin{cases} \{\mathbb{R}, \mathbb{C}, p^k \mid p \text{ prime}, k \in \mathbb{Z}_{>0}\}, & (NF) \\ \{r^k \mid k \in \mathbb{Z}_{>0}\} & (FF) \end{cases}$$

and A_f its subset of finite parameters

$$\left\{p^k \mid p \text{ prime}, k \in \mathbb{Z}_{>0}\right\}.$$

Definition 2.4. We say that a family $\mathcal{K} = \{K_i\}$ of global fields is *asymptotically exact* if the following limit exists for any $q \in A$:

$$\phi_q := \lim_{i \to +\infty} \frac{\Phi_q(K_i)}{g_{K_i}}.$$

It is said to be asymptotically good if in addition one of the ϕ_q is nonzero, and asymptotically bad otherwise. The numbers ϕ_q are called the Tsfasman—Vlăduţ invariants of the family $\mathcal{K}$.

This definition has two origins. The first one is the information theory since the families giving good algebraic geometric codes are those for which ϕ_r exists and is big. The second one is more technical and can be seen through Weil's explicit formulae. For convenience we also put

$$\phi_\infty = \lim \frac{n_{K_i}}{g_{K_i}} = \phi_{\mathbb{R}} + 2\phi_{\mathbb{C}}.$$

Being asymptotically exact is not a restrictive condition. To be precise:

Proposition 2.5. 1) *Any family of global fields contains an asymptotically exact subfamily.*

2) *Any tower of global fields is asymptotically exact and the ϕ_q's depend only on the limit.*

We can thus define the Tsfasman—Vlăduţ invariants of an infinite global fields $\mathcal{K}$ as the invariants of any tower having limit $\mathcal{K}$. From now on, we only consider asymptotically exact families, since they provide natural framework for asymptotic considerations. One of the problems of the asymptotic theory is to understand the set of possible $\{\phi_q\}$. In the next propositions we describe some the general properties of the $\{\phi_q\}$. Let us start with the basic inequalities:

Theorem 2.6 (Tsfasman—Vlăduţ). *For any asymptotically exact family of global fields, the following inequalities hold:*

$$(NF-GRH)\ \sum_q \frac{\phi_q \log q}{\sqrt{q}-1} + (\log\sqrt{8\pi} + \frac{\pi}{4} + \frac{\gamma}{2})\phi_{\mathbb{R}} + (\log 8\pi + \gamma)\phi_{\mathbb{C}} \leqslant 1,$$

$$(NF)\ \sum_q \frac{\phi_q \log q}{q-1} + (\log 2\sqrt{\pi} + \frac{\gamma}{2})\phi_{\mathbb{R}} + (\log 2\pi + \gamma)\phi_{\mathbb{C}} \leqslant 1,$$

$$(FF)\ \sum_{m=1}^{\infty} \frac{m\phi_{r^m}}{r^{\frac{m}{2}}-1} \leqslant 1,$$

where γ is the Euler constant.

This result is central in what follows. For instance, it is used to show the convergence of the limit zeta function associated to the family. It is proven using the Weil explicit formulae, the effective Chebotarev density theorem for number fields and the Riemann hypothesis for function fields.

In the case of towers of number fields (and of function fields if we consider suitable quantities), the degree of the extension gives an upper bound for the number of places above a prime number p:

Proposition 2.7. *For an asymptotically exact family of number fields and any prime number p the following inequality holds:*

$$\sum_{m=1}^{+\infty} m\phi_{p^m} \leqslant \phi_{\mathbb{R}} + 2\phi_{\mathbb{C}}.$$

Let us finally define the deficiency $\delta_{\mathcal{K}}$ of an asymptotically exact family $\mathcal{K} = \{K_i\}$ of global fields as the difference between the two sides of the basic inequalities under GRH:

$$(NF)\quad \delta_{\mathcal{K}} = 1 - \sum_q \frac{\phi_q \log q}{\sqrt{q}-1} - (\log\sqrt{8\pi} + \frac{\pi}{4} + \frac{\gamma}{2})\phi_{\mathbb{R}} - (\log 8\pi + \gamma)\phi_{\mathbb{C}}$$

and

$$(FF)\quad \delta_{\mathcal{K}} = 1 - \sum_{m=1}^{\infty} \frac{m\phi_{r^m}}{r^{\frac{m}{2}}-1}.$$

A remarkable fact is that the deficiency of infinite global fields is increasing with respect to the inclusion (see [25]): $\mathcal{K} \subset \mathcal{L}$ implies $\delta_{\mathcal{K}} \leqslant \delta_{\mathcal{L}}$. One knows that fields of zero deficiency exist in the function fields case (c.f. Section 4). Such infinite global fields are called optimal, and they are of particular interest for the information theory.

2.2. Ramification, prime decomposition and invariants

The precise statements and proofs of the results from this subsection can be found in [7] and [25]. The Tsfasman—Vlăduţ invariants of infinite global fields contain information on the ramification and the decomposition of places in these fields. Indeed, one sees from Hurwitz genus formula that any finitely ramified and tamely ramified tower of number fields is asymptotically good (because it has bounded root discriminant). For function fields, we have to ask in addition for the existence of a split place. It is not excluded that there exists an asymptotically good infinite global field with infinitely many ramified places and no split place, but no examples have been found so far. In the case of function fields, A. Garcia and H. Stichtenoth provided a widely ramified optimal tower and an everywhere ramified tower of function fields with bounded g/n is constructed in [4]. Unfortunately, we do not know anything similar for number fields.

In general, we expect asymptotically good towers to have very little ramification and some split places. The next question, first raised by Y. Ihara, is how many places split completely in a tower $\mathcal{K}$ of global field. It follows from the Chebotarev density theorem that the set of completely split places has in general a zero analytic density, that is

$$\lim_{s\to 1^+} \frac{\sum\limits_{\mathfrak{p}\in D} \mathrm{N}\mathfrak{p}^{-s}}{\sum\limits_{\mathfrak{p}\in \mathrm{Pl}_f(\mathcal{Q})} \mathrm{N}\mathfrak{p}^{-s}} = 0,$$

where D is the set of places of $\mathcal{Q}$ that split completely in $\mathcal{K}/\mathcal{Q}$. In the case of asymptotically good fields, $\sum\limits_{\mathfrak{p}\in D} \mathrm{N}\mathfrak{p}^{-1}$ is even bounded. However, in the case of asymptotically bad fields, the numerator can have an infinite limit whereas the ramification locus is very small (but infinite). We refer the reader to [25] for a more detailed treatment of the above questions.

3. Generalized Brauer—Siegel theorem and limit zeta functions

3.1. Generalizations of the Brauer—Siegel theorem

Now we turn our attention to the Brauer—Siegel theorem. The in-depth study of mathematical tools involved in it leads to an important notion of limit zeta functions which plays a key role in the study of asymptotic problems.

While looking at the statement of the Brauer—Siegel theorem (Theorem 1.1) one immediately asks a question whether the two conditions present in it are indeed necessary. It is a right guess that the second condition involving normality is technical in its nature (though getting rid of it would be a breakthrough in the analytic number theory since it is related to the so-called Siegel zeroes of zeta functions — the real zeroes which lie abnormally close to $s=1$; of course, presumably they do not exist). The second condition $n_K/\log\sqrt{|D_K|}\to 0$ looks much trickier. Using the inequalities from Proposition 2.7 it is immediate that this condition is equivalent to the fact that the family we consider is asymptotically bad.

A fundamental theorem of M. Tsfasman and S. Vlăduţ from [40] allows both to treat the asymptotically good case of the Brauer—Siegel theorem and to relax the second condition. We formulate it together with a complementary result by A. Zykin [45] which relaxes the second condition in the asymptotically bad case. Before stating the result we give the following definition:

Definition 3.1. We say that a number field K is almost normal if there exists a tower

$$K = K_n \supset \cdots \supset K_1 \supset K_0 = \mathbb{Q},$$

where each step K_i/K_{i-1} is normal.

Theorem 3.2 (Tsfasman—Vlăduţ—Zykin). *Assume that for an asymptotically exact family of number fields $\{K_i\}$ either GRH holds or all the fields K_i are almost normal. Then we have:*

$$\lim_{i\to\infty}\frac{\log(h_{K_i}R_{K_i})}{g_{K_i}} = 1+\sum_q \phi_q \log\frac{q}{q-1} - \phi_{\mathbb{R}}\log 2 - \phi_{\mathbb{C}}\log 2\pi,$$

the sum being taken over all prime powers q.

For an asymptotically bad family of number fields we have $\phi_{\mathbb{R}}=0$ and $\phi_{\mathbb{C}}=0$ as well as $\phi_q=0$ for all prime powers q, so the conclusion of the theorem takes the form of that of the classical Brauer—Siegel theorem. However, there are examples of families of number fields where the right hand side of the equality in the theorem is either strictly less or strictly greater than one (see [40]). Let us mention one particularly nice corollary of the generalized Brauer—Siegel theorem due to M. Tsfasman and S. Vlăduţ: a bound on the regulators that improves Zimmert's bound (see [44], his bound can be written as $\liminf\dfrac{\log R_{K_i}}{g_{K_i}}\geqslant(\log 2+\gamma)\phi_{\mathbb{R}}+2\gamma\phi_{\mathbb{C}})$.

Theorem 3.3 (Tsfasman—Vlăduţ). *For a family of almost normal number fields $\{K_i\}$ (or any number fields under the assumption of GRH)* *we have*

$$\liminf \frac{\log R_{K_i}}{g_{K_i}} \geqslant (\log \sqrt{\pi e} + \gamma/2)\phi_{\mathbb{R}} + (\log 2 + \gamma)\phi_{\mathbb{C}}.$$

The proof of this bound is far from being trivial, it can be found in [40].

The function field version of the Brauer—Siegel theorem is both easier to prove and requires no supplementary conditions (like normality or GRH). In fact, it was obtained before the corresponding theorem for number fields and allowed to guess what the result for number fields should be (for a proof see [36] or [39]).

Theorem 3.4 (Tsfasman—Vlăduţ). *For an asymptotically exact family of smooth projective curves $\{X_i\}$ over a finite field $\mathbb{F}_r$ we have:*

$$\lim_{i\to\infty} \frac{\log h_i}{g_i} = \log r + \sum_{f=1}^{\infty} \phi_{r^f} \log \frac{r^f}{r^f - 1},$$

where $h_i = h(X_i) = |(\operatorname{Jac} X_i)(\mathbb{F}_r)|$ is the cardinality of the Jacobian of X_i over $\mathbb{F}_r$.

Let $\varkappa_K = \operatorname{Res}_{s=1} \zeta_K(s)$ be the residue of the Dedekind zeta function $\zeta_K(s) = \prod\limits_q (1 - q^{-s})^{-\Phi_q(K)}$ of the field K at $s = 1$. Using the residue formula (see [5, Chapter VIII] and [9, Chapter III])

$$\varkappa_K = \frac{2^{\Phi_{\mathbb{R}}(K)}(2\pi)^{\Phi_{\mathbb{C}}(K)} h_K R_K}{w_K \sqrt{|D_K|}} \qquad \text{(NF case);}$$

$$\varkappa_K = \frac{h_K r^g}{(r-1)\log r} \qquad \text{(FF case)}$$

(here w_K is the number of roots of unity in K) one can see that the question about the behaviour of the ratio from the Brauer—Siegel theorem is reduced to the corresponding question for $\varkappa_K$. To put it into a more general framework, we first seek an interpretation of the arithmetic quantities we would like to study in terms of special values of certain zeta functions, then we study the behaviour of these special values in families using analytic methods. We will see in Section 5 another applications of this principle. One also notices that this reduction step explains the appearance of the GRH in the statement of the Brauer—Siegel theorem.

Let us formulate yet another version of the generalized Brauer—Siegel theorem proven by Lebacque in [23, Theorem 7]. It has the advantage of being explicit with respect to the error terms, thus giving information about the Brauer—Siegel ratio on the “finite level”.

Theorem 3.5 (Lebacque). *Let K be a global field. Then*

(i) *in the function field case*

$$\log(\varkappa_K \log r) = \sum_{f=1}^{N} \Phi_{r^f} \log \frac{r^f}{r^f - 1} - \log N - \gamma + O\Big(\frac{g_K}{Nr^{N/2}}\Big) + O\Big(\frac{1}{N}\Big);$$

(ii) *in the number field case assuming GRH*

$$\log \varkappa_K = \sum_{q \leqslant x} \Phi_q \log \frac{q}{q-1} - \log\log x - \gamma + O\Big(\frac{n_K \log x}{\sqrt{x}}\Big) + O\Big(\frac{g_K}{\sqrt{x}}\Big),$$

where $\gamma = 0.577\ldots$ is the Euler constant. The constants in O are absolute and effectively computable (and, in fact, not very big).

This theorem can also be regarded as a generalization of the Mertens theorem (see [23]). A slight improvement of the error term (as before, assuming GRH) was obtained in [26]. An unconditional number field version of this result is also available but is a little more difficult to state ([23, Theorem 6]). We should also note that Lebacque's approach leads to a unified proof of the asymptotically bad and asymptotically good cases of Theorem 3.2 with or without the assumption of GRH.

3.2. Limit zeta functions

For the moment the asymptotic theory of global fields looks like a collection of similar but not directly related results. The situation is clarified immensely by means of the introduction of limit zeta functions.

Definition 3.6. The limit zeta function of an asymptotically exact family of global fields $\mathcal{K} = \{K_i\}$ is defined as

$$\zeta_{\mathcal{K}}(s) = \prod_q (1 - q^{-s})^{-\phi_q(\mathcal{K})},$$

the product being taken over all prime powers in the number field case and over prime powers of the form $q = r^f$ in the case of curves over $\mathbb{F}_r$.

The basic inequalities from Theorem 2.6 give the convergence of the above infinite product for $\operatorname{Re} s \geqslant \frac{1}{2}$ with the assumption of GRH and for $\operatorname{Re} s \geqslant 1$ without it (in particular, in the function field case the infinite product converges for $\operatorname{Re} s \geqslant \frac{1}{2}$). In fact, the basic inequalities themselves can be restated in terms of the values of limit zeta functions. To formulate them we introduce the completed limit zeta function:

$$\tilde{\zeta}_{\mathcal{K}}(s) = e^s 2^{-\phi_{\mathbb{R}}} \pi^{-s\phi_{\mathbb{R}}/2} (2\pi)^{-s\phi_{\mathbb{C}}} \Gamma\Big(\frac{s}{2}\Big)^{\phi_{\mathbb{R}}} \Gamma(s)^{\phi_{\mathbb{C}}} \zeta_{\mathcal{K}}(s) \qquad \text{(NF case)};$$

$$\tilde{\zeta}_{\mathcal{K}}(s) = r^s \zeta_{\mathcal{K}}(s) \qquad \text{(FF case)}.$$

Let $\tilde{\xi}_{\mathcal{K}}(s)=\tilde{\zeta}'_{\mathcal{K}}(s)/\tilde{\zeta}_{\mathcal{K}}(s)$ be the logarithmic derivative of the completed limit zeta function. Then the basic inequalities from Section 2 take the following form:

Theorem 3.7 (Basic inequalities). *For an asymptotically exact family of global fields $\mathcal{K}=\{K_i\}$ we have $\tilde{\xi}_{\mathcal{K}}\left(\frac{1}{2}\right)\geqslant 0$ in the function field case and assuming GRH in the number field case and $\tilde{\xi}_{\mathcal{K}}(1)\geqslant 0$ without the assumption of GRH.*

Let us give an interesting interpretation of the deficiency in terms of the distribution of zeroes of zeta functions on the critical line. In fact, the results we are going to state are interesting on their own. To a global field K we associate the counting measure $\Delta_K=\frac{1}{g_K}\sum\limits_{\rho}\delta_{t(\rho)}$, where $t(\rho)=\operatorname{Im}\rho$ in the number field case and $t(\rho)=\frac{1}{\log r}\operatorname{Im}\rho$ in the function case; the sum is taken over all zeroes ρ of $\zeta_K(s)$ in the number field case and over all zeroes ρ of $\zeta_K(s)$ with $t(\rho)\in(-\pi,\pi]$ in the function field case (in the case of function fields $\zeta_K(s)$ is periodic with the period equal to $2\pi/\log r$), δ_t is the Dirac (atomic) measure at t. Thus we get a measure on $\mathbb{R}$ in the number field case and on $\mathbb{R}/\mathbb{Z}$ in the function field case. The asymptotic behaviour of Δ_K was first considered by Lang [4] in the asymptotically bad case. The following result is proven in [40, Theorem 5.2] and [39, Theorem 2.1].

Theorem 3.8 (Tsfasman—Vlăduţ). *For an asymptotically exact family of global fields $\mathcal{K}=\{K_i\}$, assuming GRH, the limit $\lim\limits_{i\to\infty}\Delta_{K_i}$ exists in an appropriate space of measures (to be precise, in the space of measures of slow growth on $\mathbb{R}$ in the NF case,and in the space of measures on $\mathbb{R}/\mathbb{Z}$ in the FF case). Moreover, the limit is a measure with continuous density*

$$M_{\mathcal{K}}(t)=\operatorname{Re}\tilde{\xi}_{\mathcal{K}}\left(\frac{1}{2}+it\right).$$

Of course, the expression for $M_{\mathcal{K}}(t)$ can be written explicitly using the invariants ϕ_q. Let us note two important corollaries of the theorem. First, we get an interpretation for the deficiency

$$\delta_K=\tilde{\xi}_{\mathcal{K}}\left(\frac{1}{2}\right)=M_{\mathcal{K}}(0)$$

as the asymptotic number of zeroes of $\zeta_{K_i}(s)$ accumulating at $s=\frac{1}{2}$. Second, the theorem shows that for any family of number fields zeroes of their zeta functions get arbitrarily close to $s=\frac{1}{2}$ (and, in a sense, we even know the rate at which zeroes of $\zeta_{K_i}(s)$ approach to this point).

3.3. Limit zeta functions and Brauer—Siegel type results

Let us turn our attention to the Brauer—Siegel type results. The formulae from Theorems 3.2 and 3.4 can be rewritten as

$$\lim_{i\to\infty} \frac{\log \varkappa_{K_i}}{g_{K_i}} = \log \zeta_{\mathcal{K}}(1).$$

Furthermore, using the absolute and uniform convergence of infinite products for zeta functions for $\operatorname{Re} s > 1$, Tsfasman and Vlăduţ prove in [40, Proposition 4.2] that for $\operatorname{Re} s > 1$ the equality

$$\lim_{i\to\infty} \frac{\log \zeta_{K_i}(s)}{g_{K_i}} = \log \zeta_{\mathcal{K}}(s)$$

holds. In fact, this equality remains valid for $\operatorname{Re} s < 1$ (at least if we assume GRH in the number field case). The proof of the next theorem can be found in [47] in the number field case and in [48] in the function field case (where the same problem is treated in a broader context).

Theorem 3.9 (Zykin). *For an asymptotically exact family of global fields $\mathcal{K} = \{K_i\}$ for $\operatorname{Re} s > \frac{1}{2}$ we have*

$$\lim_{i\to\infty} \frac{\log((s-1)\zeta_{K_i}(s))}{g_{K_i}} = \log \zeta_{\mathcal{K}}(s) \qquad (\textit{NF case assuming GRH});$$

$$\lim_{i\to\infty} \frac{\log((r^s-1)\zeta_{K_i}(s))}{g_{K_i}} = \log \zeta_{\mathcal{K}}(s) \qquad (\textit{FF case}).$$

The convergence is uniform on compact subsets of the half-plane $\left\{s \mid \operatorname{Re} s > \frac{1}{2}\right\}$.

The case $s=1$ of theorem 3.9 is equivalent to the Brauer—Siegel theorem and current techniques does not allow to treat it in full generality without the assumption of GRH. Thus getting unconditional results similar to Theorem 3.9 looks inaccessible at the moment. The analogue of the above result for $s = \frac{1}{2}$ is considerably weaker and one has only an upper bound:

Theorem 3.10 (Zykin). *Let ρ_{K_i} be the first non-zero coefficient in the Taylor series expansion of $\zeta_{K_i}(s)$ at $s = \frac{1}{2}$, i. e.*

$$\zeta_{K_i}(s) = \rho_{K_i}\Big(s - \frac{1}{2}\Big)^{r_{K_i}} + o\Big(\Big(s - \frac{1}{2}\Big)^{r_{K_i}}\Big).$$

Then in the function field case or in the number field case assuming that GRH is true, for any asymptotically exact family of global fields $\mathcal{K} = \{K_i\}$

the following inequality holds:

$$\limsup_{i\to\infty} \frac{\log|\rho_{K_i}|}{g_{K_i}} \leqslant \log\zeta_{\mathcal{K}}\Big(\frac{1}{2}\Big).$$

The interest in the study of the asymptotic behaviour of zeta functions at $s=\frac{1}{2}$ is partly motivated by the corresponding problem for L-functions of elliptic curves over global fields, where this value is related to deep arithmetic invariants of the elliptic curves via the Birch—Swinnerton-Dyer conjecture. We refer the reader to Section 5 for more details. The question whether the equality holds in Theorem 3.10 is rather delicate. It is related to the so called low-lying zeroes of zeta functions, that is the zeroes of $\zeta_K(s)$ having small imaginary part compared to g_K. It might well happen that the equality

$$\lim_{i\to\infty} \frac{\log|\rho_{K_i}|}{g_{K_i}} = \log\zeta_{\mathcal{K}}\Big(\frac{1}{2}\Big)$$

does not hold for all asymptotically exact families $\mathcal{K}=\{K_i\}$ since the behaviour of low-lying zeroes is known to be rather random. Nevertheless, it might hold for “most” families (whatever it might mean).

To illustrate how hard the problem may be, let us remark that Iwaniec and Sarnak studied a similar question for the central values of L-functions of Dirichlet characters [16] and modular forms [17]. They manage to prove that there exists a positive proportion of Dirichlet characters (modular forms) for which the logarithm of the central value of the corresponding L-functions divided by the logarithm of the analytic conductor tends to zero. The techniques of the evaluation of mollified moments used in these papers are rather involved. We also note that, to our knowledge, there has been no investigation of low-lying zeroes of L-functions of growing degree. It seems that the analogous problem in the function field case has neither been very well studied.

Let us indicate that the corresponding question for the logarithmic derivatives of zeta functions has a negative answer. Indeed, the functional equation implies that $\lim\limits_{i\to\infty} \dfrac{\zeta'_{K_i}(1/2)}{\zeta_{K_i}(1/2)} = 1$ for any family of function fields K_i. However, the logarithmic derivative of the limit zeta function $\zeta_{\mathcal{K}}(s)$ at $s=\frac{1}{2}$ equals one only for asymptotically optimal families (c.f. Theorem 3.7).

As a corollary of Theorem 3.9 one can obtain a result on the asymptotic behaviour of Euler—Kronecker constants.

Definition 3.11. The Euler—Kronecker constant of a global field K is defined as $\gamma_K = \frac{c_0(K)}{c_{-1}(K)}$, where $\zeta_K(s) = c_{-1}(K)(s-1)^{-1} + c_0(K) + O(s-1)$.

In [14] Y. Ihara made an extensive study of the Euler—Kronecker constants of global fields, in particular, he obtained an asymptotic formula for their behaviour in families of curves over finite fields. A complementary result in the number field setting was obtain in [47] as a corollary of Theorem 3.9. In fact Theorem 3.9 gives that in asymptotically exact families the coefficients of the Laurant series at $s=1$ of the logarithmic derivatives $\zeta'_{K_i}(s)/\zeta_{K_i}(s)$ tend to the corresponding coefficients of the Laurant series expansion of the logarithmic derivative of the limit zeta function. For zeroes coefficient this becomes:

Corollary 3.12 (Ihara—Zykin). *Assuming GRH in the number field case and unconditionally in the function field case, for any asymptotically exact family of global fields $\{K_i\}$ we have*

$$\lim_{i\to\infty} \frac{\gamma_{K_i}}{g_{K_i}} = -\sum_q \phi_q \frac{\log q}{q-1}.$$

For the sake of completeness let us mention an explicit analogue of Theorem 3.9 obtained in [26]:

Theorem 3.13 (Lebacque—Zykin). *For any global field K, any integer $N \geqslant 10$ and any $\varepsilon = \varepsilon_0 + i\varepsilon_1$ such that $\varepsilon_0 = \operatorname{Re}\varepsilon > 0$ we have*

(i) *in the function field case:*

$$\sum_{f=1}^{N} \frac{f\Phi_{r^f}}{r^{\left(\frac{1}{2}+\varepsilon\right)f}-1} + \frac{1}{\log r}\cdot Z_K\Big(\frac{1}{2}+\varepsilon\Big) + \frac{1}{r^{-\frac{1}{2}+\varepsilon}-1} = O\Big(\frac{g_K}{r^{\varepsilon_0 N}}\Big(1+\frac{1}{\varepsilon_0}\Big)\Big) + O\big(r^{\frac{N}{2}}\big);$$

(ii) *and in the number field case assuming GRH:*

$$\sum_{q\leqslant N} \frac{\Phi_q \log q}{q^{\frac{1}{2}+\varepsilon}-1} + Z_K\Big(\frac{1}{2}+\varepsilon\Big) + \frac{1}{\varepsilon-\frac{1}{2}} =$$
$$= O\Big(\frac{|\varepsilon|^4+|\varepsilon|}{\varepsilon_0^2}(g_K + n_K\log N)\frac{\log^2 N}{N^{\varepsilon_0}}\Big) + O\big(\sqrt{N}\big).$$

3.4. Some other topics related to limit zeta functions

Let us finally state some related results on the asymptotic properties of the coefficients of zeta functions. For the moment they are only available in the function field case (see [39]). Let $K/\mathbb{F}_r(t)$ be a function field and let $\zeta_K(s) = \sum_{m=1}^{\infty} D_m r^{-ms}$ be the Dirichlet series expansion of the zeta function of K. One knows that D_m is equal to the number of effective divisors of degree m on the corresponding curve. We have the following results on the asymptotic behaviour of D_m:

Theorem 3.14 (Tsfasman—Vlăduţ). *For an asymptotically exact family of function fields $\mathcal{K}=\{K_i\}$ and any real $\mu>0$ we have*

$$\lim_{i\to\infty}\frac{\log D_{[\mu g]}(K_i)}{g_{K_i}}=\min_{s\geqslant 1}(\mu s\log q+\log\zeta_{\mathcal{K}}(s)).$$

Moreover, the minimum can be evaluated explicitly via ϕ_q (c.f. [39, Proposition 4.1]).

Theorem 3.15 (Tsfasman—Vlăduţ). *For an asymptotically exact family of function fields $\mathcal{K}=\{K_i\}$, any $\varepsilon>0$ and any m such that $\frac{D_m}{g}\geqslant\mu_1+\varepsilon$ we have*

$$\frac{\log D_m(K_i)}{h_{K_i}}=\frac{q^{m-g+1}}{q-1}(1+o(1))$$

for $g\to\infty$, $o(1)$ being uniform in m. Here μ_1 is the largest of the two roots of the equation

$$\frac{\mu}{2}+\mu\log_r\frac{\mu}{2}+(2-\mu)\log_r\Big(1-\frac{\mu}{2}\Big)=-2\log_r\zeta_{\mathcal{K}}(1).$$

We should note that $o(1)$ from Theorem 3.15 is additive whereas most of the previous results were estimates of multiplicative type (they contained logarithms of the quantities in question). It would be interesting to know whether there exist analogues of the above results in the number field case.

Let us conclude by refering the reader to the Section 6 of [40] for a list of open questions.

4. Examples

4.1. Towers of modular curves

Let us begin with the examples of asymptotically optimal families of curves over finite fields coming from towers of modular curves. The first constructions were carried out by Ihara [12], Tsfasman—Vlăduţ—Zink [42]. The research in this direction was continued by N. Elkies and many others. Let us describe several constructions.

4.1.1. *Classical modular curves*

Let us start with the construction of towers of modular curves which leads to asymptotically optimal infinite function fields. For further information, we refer the reader to [38, Chapter 4]. It is well known that the modular group $\Gamma(1)=\mathrm{PSL}_2(\mathbb{Z})$ acts on the Poincaré upper half-plane $\mathfrak{h}$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \dfrac{az+b}{cz+d}$. We fix a positive integer N and we define the principal congruence subgroup of level N by

$$\Gamma(N) = \left\{\gamma \in \Gamma(1) \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathrm{mod}\ N\right\}.$$

$\Gamma(N) \lhd \Gamma(1)$ and $\Gamma(1)/\Gamma(N)$ is isomorphic to $PSL_2(\mathbb{Z}/N\mathbb{Z})$. In particular,

$$[\Gamma(1) : \Gamma(N)] = \begin{cases} \dfrac{N^3}{2} \displaystyle\prod_{\ell \mid N} \left(1-\ell^{-2}\right) & \text{if } N \geqslant 3 \\ 6 & \text{if } N = 2. \end{cases}$$

We also put $\Gamma_0(N)=\left\{\gamma\in\Gamma(1) \mid \gamma\equiv\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\mathrm{mod}\ N\right\}$, so that $\Gamma(N)\subset\Gamma_0(N)$. We have $[\Gamma(1):\Gamma_0(N)]=N\prod_{\ell \mid N}(1-\ell^{-1})$.

Let now Γ be a congruence subgroup, that is, any subgroup of $\Gamma(1)$ containing $\Gamma(N)$. The most important case for us is $\Gamma=\Gamma(N)$ or $\Gamma_0(N)$. The set $Y_\Gamma=\Gamma\backslash\mathfrak{h}$ is equipped with an analytic structure, but is not compact. To compactify it we add points at infinity (named cusps): $\Gamma(1)$ acts naturally on $\mathbb{P}^1(\mathbb{Q})$ and we put $X_\Gamma=(\Gamma\backslash\mathfrak{h})\cup(\Gamma\backslash\mathbb{P}^1(\mathbb{Q}))$. This way it becomes a connected Riemann surface called modular curve. We let $X(N)=X_{\Gamma(N)}$, $X_0(N)=X_{\Gamma_0(N)}$, $Y(N)=Y_{\Gamma(N)}$ and $Y_0(N)=X_{\Gamma_0(N)}$.

If $\Gamma'\subset\Gamma\subset\Gamma(1)$, there is a natural projection from $X_{\Gamma'}\to X_\Gamma$, which allows us to compute the genus of the modular curve using the covering (the function j is in fact the j-invariant of the elliptic curve $\mathbb{C}/(\mathbb{Z}+z\mathbb{Z})$):

$$X_\Gamma \longrightarrow X_{\Gamma(1)} \xrightarrow[j]{\sim} \mathbb{P}^1(\mathbb{C})$$

via the Hurwitz formula. For instance,

$$g_{X(N)} = 1 + \frac{(N-6)[\Gamma(1) : \Gamma(N)]}{12N}.$$

It can be shown that $Y(1)$ classifies isomorphism classes of complex elliptic curves and that $Y_0(N)$ classifies pairs (E, C_N), E being a complex elliptic curve and C_N being a cyclic subgroup of E of order N.

Now, to construct towers of curves defined over finite fields, we need to take reductions of our modular curves modulo primes. If S is a scheme and $E\to S$ is an elliptic curve, the set of sections $E(S)$ is an abelian group. Let $E_N(S)$ denote the points of order dividing N in $E(S)$. We call a level

N structure an isomorphism $\alpha_N : E_N(S) \to (\mathbb{Z}/N\mathbb{Z})^2$. One can prove that there exists a smooth affine scheme $Y(N)$ over $\operatorname{Spec}\mathbb{Z}[1/N]$ classifying the isomorphism classes of pairs (E, α_N) consisting of an elliptic curve $E/\operatorname{Spec}\mathbb{Z}[1/N]$ together with a level N structure α_N on E. One can prove that this curve is a model of $Y(N)$ over $\operatorname{Spec}\mathbb{Z}[\zeta_N, 1/N]$, where ζ_N is a primitive N^{th}-root of 1. There is also a model of $Y_0(N)$ over $\operatorname{Spec}\mathbb{Z}[1/N]$ and this "coarse" moduli space classifies pairs consisting of an elliptic curve together with a cyclic subgroup of order N. Models for $X(N)$ and $X_0(N)$ can also be obtained in such a way that they become compatible with those for $Y(N)$ and $Y_0(N)$. These curves have good reduction over any prime ideal not dividing N. Moreover, the curve $X_0(N)$ can be defined over $\mathbb{Q}$ and has good reduction at any prime number not dividing N. Let p be such prime. We denote by $C_{0,N}$ the curve over $\mathbb{F}_{p^2}$ obtained by reduction of $X_0(N)$ mod p. The curve $X(N)$ can be defined over the quadratic subfield of $\mathbb{Q}(\zeta_N)$ and has good reduction at all the primes not dividing N. Let C_N be the reduction of $X(N)$ at a prime, i. e. a curve over $\mathbb{F}_{p^2}$. One can see that the genus of $X_0(N)$ and of $X(N)$ is preserved under reduction. The points of these curves corresponding to supersingular elliptic curves are $\mathbb{F}_{p^2}$-rational and there are $\dfrac{[\Gamma(1):\Gamma(N)]}{12}(p-1)$ of them on C_N. This leads to the following theorem:

Theorem 4.1 (Ihara, Tsfasman—Vlăduţ—Zink). *Let ℓ be a prime number not equal to p. The families $\{C_{\ell^n}\}$ and $\{C_{0,\ell^n}\}$ satisfy $\phi_{p^2} = p-1$ and therefore are asymptotically optimal.*

Note that the result for C_{0,ℓ^n} can be deduced immediately from the corresponding result for C_{ℓ^n}.

4.1.2. *Shimura modular curves*

Similar results on Shimura curves allow us to construct directly asymptotically optimal families over $\mathbb{F}_r$ with $r = q^2 = p^{2m}$, p prime. To do so, following Ihara, we start with a $\mathfrak{p}$-adic field $k_{\mathfrak{p}}$ with $N(\mathfrak{p}) = q = p^m$. Let Γ be a torsion-free discrete subgroup of $G = PSL_2(\mathbb{R}) \times PSL_2(k_{\mathfrak{p}})$ with compact quotient and dense projection to each of the two components of G (such Γ's exist). Ihara proved the following results that relate the construction of optimal curves to (anabelian) class field theory, and therefore are of great interest for us:

Theorem 4.2 (Ihara [15]). *To any subgroup Γ of G with the above properties one can associate a complete smooth geometrically irreducible curve X over $\mathbb{F}_r$ of genus $\geqslant 2$, together with a set Σ consisting of $(q-1)(g-1)$ $\mathbb{F}_r$-rational points of X such that there is a canonical isomorphism (up to* *conjugacy) from the profinite completion of* Γ *to* $Gal(K^{\Sigma}/K)$ *where* K^{Σ} *denotes the maximal unramified Galois extension of the function field* K *of* X *in which all the places corresponding to the points of* Σ *are completely split.*

An easy computation leads to the following result:

Corollary 4.3. *For any square prime power* r*, there is a tower of curves defined over* $\mathbb{F}_r$ *with* $\phi_r = \sqrt{r} - 1$.

In fact, the elliptic modular curves $X(N)$ that we constructed in the previous section correspond to $\Gamma = PSL_2(\mathbb{Z}[1/p])$ and its principal congruence subgroups of level N.

4.1.3. *Drinfeld modular curves*

The applicability of Drinfeld modular curves to the problem of construction of optimal curves has been known since late 80's. The results we are going to discuss next can be found in [38].

Let L be a field of characteristic p and let $L\{\tau\}$ denote the ring of non-commutative polynomials in τ, consisting of expressions of the form $\sum_{i=0}^{n} a_i \tau^i$, $a_i \in L$, with multiplication satisfying $\tau \cdot a = a^p \cdot \tau$ for any $a \in L$. Let $A = \mathbb{F}_r[T]$.

A Drinfeld module is an $\mathbb{F}_r$-homomorphism $\phi : A \to L\{\tau\},\ a \mapsto \phi_a$ satisfying a few technical conditions. Let γ be the map $\gamma : A \to L$ sending $a \in A$ to the term of ϕ_a of degree zero. Notice that ϕ is determined by ϕ_T and γ by $\gamma(T)$. We consider only Drinfeld modules of rank 2 that is we assume that ϕ_T is a polynomial in τ of degree 2 and we put $\phi_T = \gamma(T) + g\tau + \Delta\tau^2$ $(\Delta \neq 0)$. More generally, one can define Drinfeld modules over any A-scheme S.

Just as in the classical case, given a proper ideal I of A, one can define a level I structure on ϕ. There is an affine scheme $M(I)$ of finite type over A that parametrizes pairs (ϕ, λ), where ϕ is a Drinfeld module over S and λ is a level I structure. The scheme $M(I)$ has a canonical compactification: there exists a unique scheme $\overline{M(I)}$ containing $M(I)$ as an open dense subscheme, whose fibres over Spec $A[I^{-1}]$ are smooth complete curves. The group $\mathrm{GL}_2(A/I)$ acts naturally on $M(I)$ by operating on the structures of level I and this action extends to $\overline{M(I)}$.

From now on, let I be a prime ideal generated by a polynomial of degree m prime to $q-1$. Now, consider the smooth complete (reducible) curve $X(I) = \overline{M(I)} \otimes_A \mathbb{F}_q$ over $\mathbb{F}_q$. Note that the A-algebra structure on $\mathbb{F}_q$ is obtained through the reductionmod T. Consider the subgroup

$$\Gamma_0(I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A) \mid c \in I \right\}$$

and let $\overline{\Gamma_0(I)}$ be the image of this subgroup in $\mathrm{GL}_2(A/I)$. Finally, we consider the smooth complete absolutely irreducible curve $X_0(I)=X(I)/\overline{\Gamma_0(I)}$. The image of $\overline{M(I)}-M(I)$ in $X_0(I)$ consists of two $\mathbb{F}_q$-rational points. Moreover, the following result holds.

Theorem 4.4. *The family $\{X_0(I)\}$, where I is a prime ideal of A generated by a polynomial of degree prime to $q-1$, is an asymptotically exact family of curves defined over $\mathbb{F}_q$, satisfying $\phi_{q^2}=q-1$ and thus is optimal.*

Moreover, N. Elkies proved in [6] that the family of curves $\dot{X}_0(T^n)$ which parametrizes normalized Drinfeld modules ($\gamma(T)=1, \Delta=-1$) with a level T^n structure is asymptotically optimal. He also related it to the explicit towers of Garcia and Stichtenoth discussed in the next subsection.

4.2. Explicit towers

In the last fifteen years, Garcia, Stichtenoth and many others managed to construct asymptotically good towers explicitely in a recursive way. Their interest comes from coding theory for such towers provide asymptotically good codes via the construction of Goppa. Let us give an example of such explicit towers.

Theorem 4.5 (Garcia—Stichtenoth). *Let $r=q^2$ be a prime power. The tower $\{F_n\}$ defined recursively starting from the rational function field $F_0=\mathbb{F}_r(x_0)$ using the relations $F_{n+1}=F_n(x_{n+1})$, where*

$$x_{n+1}^q + x_{n+1} = \frac{x_n^q}{x_n^{q-1}+1},$$

satisfies $\phi_r=\sqrt{r}-1$ and thus is optimal.

If the cardinality of the ground field is not a square no towers with $\phi_r=\sqrt{r}-1$ are known. However, there exist optimal towers in the sense that they have zero deficiency. Such towers can be constructed starting from an explicit tower over a bigger field using a descent argument (see Ballet—Rolland [2] for the details) or using modular towers.

Let us now say a word about Elkies modularity conjecture. Elkies' work shows that most of the recursive examples of Garcia and Stichtenoth can be obtained by finding equations for suitable modular towers. This made him formulate the following conjecture:

Conjecture 4.6 (Elkies). *Any asymptotically optimal tower is modular.*

Finally, let us note that there are other interesting constructions leading to explicit asymptotically good towers of function fields. As an example we mention the paper [1] by P. Beelen and I. Bouw who use Fuchsian differential equations to produce optimal towers over $\mathbb{F}_{q^2}$.

4.3. Classfield towers

As it was said in Section 2, tamely ramified infinite extensions of global fields with finitely many ramified places and with completely split places give examples of asymptotically good towers. Given a global field K, it is natural to consider the maximal extension of K unramified outside a finite set of places S, in which places from a set T are completely split. But these extensions are very hard to understand. The maximal ℓ-extensions are much easier to handle. These extensions are the limits of the ℓ-S-T-class field towers of K.

For a global field K, two sets of finite places S and T ($T\neq\varnothing(FF)$) of K, and a prime number ℓ, consider the maximal abelian ℓ-extension $H^T_{S,\ell}(K)$ of K, unramified outside S and in which the places from T are split (in the case of function fields the assumption on T to be non-empty is made in order to avoid infinite constant field extensions). Consider the tower recursively constructed as follows: $K_0=K$, $K_{i+1}=H^T_{S,\ell}(K_i)$. All the extensions K_i/K are Galois, and we denote by $G^T_S(K,\ell)$ the Galois group $\mathrm{Gal}\Big(\bigcup\limits_i K_i, K\Big)$. A sufficient condition for this tower to be infinite is given by the Golod—Shafarevich theorem: if G is a finite ℓ-group then $\dim_{\mathbb{F}_\ell} H^2(G,\mathbb{F}_\ell)>\frac{1}{4}\dim_{\mathbb{F}_\ell} H^1(G,\mathbb{F}_\ell)^2$. This allows to construct asymptotically good infinite global fields. The following result is at the base of many constructions of class field towers with prescribed properties:

Theorem 4.7 (Tsfasman—Vlăduţ [40] (NF), Serre [33] , Niederreiter—Xing [28] (FF))**.** *Let K/k be a cyclic extension of global fields of degree ℓ. Let $T(k)$ be a finite set of non archimedean places of k and let $T(K)$ be the set of places above $T(k)$ in K. Suppose in the function field case that* $\mathrm{GCD}\{\ell,\deg\mathfrak{p},\mathfrak{p}\in T(K)\}=1$. *Let Q be the ramification locus of K/k. Let*

$$(FF)\quad C(T,K/k) = \#T(k)+2+\delta_\ell+2\sqrt{\#T(K)+\delta_\ell},$$

$$(NF)\quad C(T,K/k) = \#T(K)-t_0+r_1+r_2+\delta_\ell+2-\rho+ \\ +2\sqrt{\#T(K)+\ell(r_1+r_2-\rho/2)+\delta_\ell},$$

where $\delta_\ell=1$ if K contains the ℓ-root of unity, and 0 otherwise, t_0 is the number of principal ideals in $T(k)$, $r_1=\Phi_{\mathbb{R}}(K)$, $r_2=\Phi_{\mathbb{C}}(K)$ and ρ is the number of real places of k which become complex in K. Suppose that $\#Q\geqslant C(T,K/k)$. Then K admits an infinite unramfied ℓ-$T(K)$-class field tower.

One can construct such cyclic extension using the Grunwald-Wang theorem (and sometimes even explicitly by hand) and deduce the following result:

Corollary 4.8 (Lebacque). *Let n be an integer and let $t_1, \dots, t_n$ be prime powers* (*NF*) (*powers of p* (*FF*)). *There exists an infinite global field* (*both in the number field and function field cases*) *such that $\phi_{t_1}, \dots, \phi_{t_n}$ are all >0.*

Another way to produce asymptotically good infinite class field towers is to use tamely ramified instead of unramified class field towers. This is the subject of [9] and [10].

The question of finding asymptotically good towers with given Tsfasman—Vlăduţ invariants equal to zero is more difficult. A related question is to find out whether an infinite global extension realizes the maximal local extension at a given prime. Using results of J. Labute [19] and A. Schmidt [31], the following theorem is proven:

Theorem 4.9 (Lebacque [24]). *Let $P=\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}\subset Pl_f(\mathcal{Q})$. Assume that for any $i=1, \dots, n$ we have n_i distinct positive integers $d_{i,1}, \dots, d_{i,n_i}$. Let $I \subset Pl_f(\mathcal{Q})$ be a finite set of finite places of $\mathcal{Q}$ such that $I \cap P = \varnothing$. There exists an infinite global field $\mathcal{K}$ such that:*

1) $I \cap \operatorname{Supp}(\mathcal{K}) = \varnothing$,

2) *For any $i=1, \dots, n$, and any $j=1, \dots, n_i$,* $\phi_{\mathfrak{p}_i, \mathrm{N}\mathfrak{p}_i^{d_{i,j}}} = \dfrac{\phi_\infty}{n_i d_{i,j}} > 0.$

3) *One can explicitly estimate ϕ_∞ and the deficiency in terms of P, I, n_i and d_{ij}.*

The $\phi_{\mathfrak{p},q}$ are invariants generalizing the classical ϕ_q: they count the asymptotic number of primes of norm q above a given prime $\mathfrak{p}$ (see [25] for a definition). In the case of $\mathbb{Q}$ they coincide with the classical ones. This extension is obtained as the compositum of a finite extension of $\mathcal{Q}$ with prescribed positive $\Phi_{\mathfrak{p}_i, \mathrm{N}\mathfrak{p}_i^{d_{i,j}}} > 0$ and an infinite class field tower $\mathcal{Q}_S^P(\ell)$ satisfying the $K(\pi, 1)$ property of A. Schmidt.

4.4. Bounds on the deficiency

We have already seen that, using towers of modular curves, one can produce infinite function fields over $\mathbb{F}_r$ with zero deficiency. If r is a square, there are even towers with $\phi_r = \sqrt{r} - 1$. In the case of number fields no zero deficiency infinite number fields are known. In fact we doubt that the class field theory (which is for now the only method to produce asymptotically good infinite number fields) can ever give such field. Let us quote here the example with the smallest known deficiency due to F. Hajir and Ch. Maire [10].

Let $k = \mathbb{Q}(\xi)$, where ξ is a root of $f(x) = x^6 + x^4 - 4x^3 - 7x^2 - x + 1$. Consider the element

$$\eta = -671\xi^5 + 467\xi^4 - 994\xi^3 + 3360\xi^2 + 2314\xi - 961 \in \mathcal{O}_k.$$

Let $K = k(\sqrt{\eta})$. F. Hajir and Ch. Maire prove using a Golod—Shafarevich like result that K admits an infinite tamely ramified tower satisfying $\delta \leqslant 0.137\ldots$.

5. Higher dimensional theory

In this section we will mostly consider the function field case since most of the results we are going to mention are unavailable in the number field case. However, we will give some references to the number field case as well.

5.1. Number of points on higher dimensional varieties

The question about the maximal number of points on curves over finite fields has been extensively studied by numerous authors. The analogous question for higher dimensional varieties has received comparatively little attention most probably due to its being significantly more difficult.

As for the curves, we have the so-called Weil bound which is in this case a famous theorem of Deligne. Similarly, this bound is not optimal and the general framework for improving it is provided by the explicit formulae. In the case of curves over $\mathbb{F}_r$ Oesterlé managed to find the best bounds available through the techniques of explicit formulae for any given $r \neq 2$ (see [33]). A decade later the case of arbitrary varieties over finite fields was treated by G. Lachaud and M. A. Tsfasman in [37] and [22]. Let us reproduce here the main results from [22]. To do so we will have to introduce some notation concerning varieties over finite fields.

Let X be a non-singular absolutely irreducible projective variety of dimension d defined over a finite field $\mathbb{F}_r$. We put $X_f = X \otimes_{\mathbb{F}_r} \mathbb{F}_{r^f}$ and $\overline{X} = X \otimes_{\mathbb{F}_r} \overline{\mathbb{F}}_r$. Let $\Phi_{r^f} = \Phi_{r^f}(X)$ be the number of points of X having degree f. Thus, for the number N_f of $\mathbb{F}_{r^f}$-points of the variety X_f we have the formula $N_f = \sum\limits_{m|f} m\Phi_{r^m}$. We denote by $b_s(X) = \dim_{\mathbb{Q}_l} H^s(\overline{X}, \mathbb{Q}_l)$ the l-adic Betti numbers of X.

The family of inequalities proven in [22] has a doubly positive sequence as a parameter. Let us introduce the corresponding notation. To a sequence of real numbers $\mathbf{v}=(v_n)_{n\geqslant 0}$ we associate the family of power series $\psi_{m,\mathbf{v}}(t)=\sum\limits_{n=1}^{\infty} v_{mn}t^n$. We denote $\psi_{\mathbf{v}}(t)=\psi_{1,\mathbf{v}}(t)$ and let $\rho_{\mathbf{v}}$ be the radius of convergence of this power series. A doubly positive sequence $\mathbf{v}$ is such a sequence that $0\leqslant v_n\leqslant v_0$ for all n, $v_0=1$ and for any $z\in\mathbb{C}$, $|z|<1$ we have $1+2\operatorname{Re}\psi_{\mathbf{v}}(t)\geqslant 0$.

We will also need the functions

$$F_{m,\mathbf{v}}(k,t)=\sum_{s=0}^{\infty}(-1)^s\psi_{m,\mathbf{v}}(r^{-ks}t)=\sum_{n=1}^{\infty}\frac{v_{mn}t^{mn}}{1+r^{-mnk}},\qquad F_{\mathbf{v}}(k,t)=F_{1,\mathbf{v}}(k,t).$$

We let $A_{\mathbf{v}}(z)=-\min\limits_{|t|=z}\operatorname{Re}\psi_{\mathbf{v}}(t)$ and denote

$$I(k)=\{i\mid 1\leqslant i\leqslant 2d-1, i\neq k, i\neq 2d-k\}$$

the set of indices. We have the following inequalities:

Theorem 5.1 (Lachaud—Tsfasman). *For any odd integer k, $1\leqslant k\leqslant d$, any doubly positive sequence $\mathbf{v}=(v_n)_{n\geqslant 0}$ with $\rho_{\mathbf{v}}>q^{k/2}$ and any $M\geqslant 1$ we have*

$$\sum_{m=1}^{M} m\Phi_{r^m}(X)\psi_{m,\mathbf{v}}(r^{-(2d-k)/2})\leqslant \psi_{\mathbf{v}}(r^{-(2d-k)/2})+\psi_{\mathbf{v}}(r^{k/2})+\frac{b_k}{2}+$$
$$+\sum_{i\ odd, i\neq k} b_iA_{\mathbf{v}}(r^{-(i-k)/2})+\sum_{i\ even} b_i\psi_{\mathbf{v}}(r^{-(i-k)/2}),$$

and

$$\sum_{m=1}^{M} m\Phi_{r^m}(X)F_{m,\mathbf{v}}(d-k,r^{-(2d-k)/2})\leqslant$$
$$\leqslant F_{\mathbf{v}}(d-k,r^{-(2d-k)/2})+F_{\mathbf{v}}(d-k,r^{k/2})+\frac{b_k}{2}+\sum_{i\in I(k)} b_iF_{\mathbf{v}}(d-k,r^{-(i-k)/2}).$$

For example, taking the second inequality with $\psi_{\mathbf{v}}(t)=\frac{t}{2}$ we get the classical Weil bound, taking the first one with $\psi_{\mathbf{v}}(t)=\frac{t}{1-t}$ we get (asymptotically) a direct generalization of the Drinfeld—Vlăduţ bounds. These inequalities are not straightforward to apply. We refer the reader to [22] for more details on how to make good choices of the doubly positive sequence. Unfortunately, in the case of dimension $d\geqslant 2$ the optimal choice of $\mathbf{v}$ is unknown.

The asymptotic versions of these inequalities can be easily deduced from Theorem 5.1 once one introduces proper definitions. For a variety X let $b(X)=\max_{i=0,\ldots,d} b_i(X)$ be the maximum of its l-adic Betti numbers.

Definition 5.2. A family of varieties $\{X_j\}$ is called asymptotically exact if the limits $\phi_{r^f}=\lim_{j\to\infty}\frac{\Phi_{r^f}(X_j)}{b(X_j)}$ and $\beta_i=\lim_{j\to\infty}\frac{b_i(X_j)}{b(X_j)}$ exist. It is asymptotically good if at least one of ϕ_{r^f} is different from zero.

We can state the following corollary of Theorem 5.1:

Corollary 5.3. *In the notation of Theorem 5.1 for an asymptotically exact family of varieties one has*

$$\sum_{m=1}^{M} m\phi_{r^m}\psi_{m,\mathbf{v}}(r^{-(2d-k)/2})\leqslant$$
$$\leqslant\frac{\beta_k}{2}+\sum_{i\ odd,i\neq k}\beta_i A_{\mathbf{v}}(r^{-(i-k)/2})+\sum_{i\ even}\beta_i\psi_{\mathbf{v}}(r^{-(i-k)/2}),$$

and

$$\sum_{m=1}^{M} m\phi_{r^m}F_{m,\mathbf{v}}(d-k,r^{-(2d-k)/2})\leqslant\frac{\beta_k}{2}+\sum_{i\in I(k)}\beta_i F_{\mathbf{v}}(d-k,r^{-(i-k)/2}).$$

Taking particular examples of the sequence $\mathbf{v}$ one gets more tractable inequalities (see [22]).

5.2. Brauer—Siegel type conjectures for abelian varieties over finite fields

One can ask about the possibility of extending the Brauer—Siegel theorem to the case of varieties over finite fields. The question is not as easy as it might seem. First, mimicking the proof of Theorem 3.4 one gets a result about the asymptotic behaviour of the residues of zeta functions of varieties at $s=d$ (see [46]). Such a result would be interesting if there was a reasonable interpretation for this residue in terms of geometric invariants of our variety.

Two other approaches were suggested by B. Kunyavskii and M. Tsfasman and by M. Hindry and A. Pacheco. Both of them have for their starting points the Birch and Swinnerton-Dyer (BSD) conjecture which expresses the value at $s=1$ of the L-function of an abelian variety in terms of certain arithmetic invariants related to this variety. However, the situation with the asymptotic behaviour of this special value of the L-functions is much less clear than before. Let us begin with the approach of Kunyavskii and Tsfasman.

Let $K/\mathbb{F}_r$ be a function field and let A/K be an abelian variety over K. We denote by $\text{Ш}_A := |\text{Ш}(A/K)|$ the order of the Shafarevich—Tate group of A, and by Reg_A the determinant of the Mordell—Weil lattice of A (see [11] for definitions). Note that in a certain sense Ш_A and Reg_A are the analogues of the class number and of the regulator respectively. Kunyavskii and Tsfasman make the following conjecture concerning families of constant abelian varieties (see [18]):

Conjecture 5.4. *Let A_0 be a fixed abelian variety over $\mathbb{F}_r$. Take an asymptotically exact family of function fieds $\mathcal{K} = \{K_i\}$ and put $A_i = A_0 \times \times_{\mathbb{F}_r} K_i$. Then*

$$\lim_{i\to\infty} \frac{\log_r(\text{Ш}_i \cdot \mathrm{Reg}_i)}{g_i} = 1 - \sum_{m=1}^{\infty} \phi_{r^m}(\mathcal{K}) \log_r \frac{|A_0(\mathbb{F}_{r^m})|}{r^m}.$$

This conjecture is actually stated as theorem in [18]. Unfortunately the change of limits in the proof given in [18] is not justified thus the proof can not be considered a valid one. In fact the flaw looks very difficult to repair as the statement of the theorem can be reduced (via a formula due to J. Milne, which gives the BSD conjecture in this case) to an equality of the type $\lim\limits_{i\to\infty} \dfrac{\log \zeta_{K_i}(s)}{g_{K_i}} = \log \zeta_{\mathcal{K}}(s)$ at a given point $s \in \mathbb{C}$ with $\mathrm{Re}\, s = \frac{1}{2}$ (in fact s belongs to a finite set of points depending on A_0). As we have already mentioned in the discussion following Theorem 3.10 this question does not look accessible at the moment.

Let us turn our attention to the approach of Hindry and Pacheco. They treat the case in some sense “orthogonal” to that of Kunyavskii and Tsfasman. Here is the conjecture they make in [11]:

Conjecture 5.5. *Consider the family $\{A_i\}$ of non-constant abelian varieties of fixed dimension over the fixed function field K. We have*

$$\lim_{i\to\infty} \frac{\log(\text{Ш}_i \cdot \mathrm{Reg}_i)}{\log H(A_i)} = 1,$$

where $H(A_i)$ is the exponential height of A_i.

Using deep arguments from the theory of abelian varieties over function fields the conjecture is reduced in [11] to the one on zeroes of L-functions of abelian varieties together with the BSD conjecture. Hindry and Pacheco are actually faced with the problem of the type discussed after Theorem 3.10, this time for abelian varieties over function fields.

The following example serves as the evidence for the last conjecture (see [11]):

Theorem 5.6 (Hindry—Pacheco). *For the family of elliptic curves E_d over $\mathbb{F}_r(t)$, where the characteristic of $\mathbb{F}_r$ is not equal to 2 or 3, defined by the equations $y^2+xy=x^3-t^d$, $d\geqslant 1$ and prime to r, the Tate—Shafarevich group Ш(E_d/K) is finite and*

$$\log(Ш_d \cdot \mathrm{Reg}_d) \sim \log H(E_d) \sim \frac{d\log r}{6}.$$

The proof of this theorem uses a deep result of Ulmer [43] who established the BSD conjecture in this case and explicitly computed the L-functions of E_d. This reduces the statement of the theorem to a an explicit (though highly non-trivial) estimate involving Jacobi sums.

The Conjectures 5.4 and 5.5 can be united (though not proved) within the general asymptotic theory of L-functions over function fields. Such a theory also explains why we get 1 as a limit in the second conjecture and a complicated expression in the first one. We will sketch some aspects of the theory in the next subsection.

The analogous problem in the number field case has also been considered [8]. Unfortunately in the number field case we do not have a single example supporting the conjecture.

5.3. Asymptotic theory of zeta and L-functions over finite fields

The proofs of the results from this subsection as well as lengthy discussions can be found in [48]. Let us first define axiomatically the class of functions we are going to work with. This resembles the so called Selberg class from the analytic number theory, but, of course the case of function fields is infinitely easier from the analytic point of view, all functions being rational (or even polynomial).

Definition 5.7. An L-function $L(s)$ over a finite field $\mathbb{F}_r$ is a holomorphic function in s such that for $u=q^{-s}$ the function $\mathscr{L}(u)=L(s)$ is a polynomial with real coefficients, $\mathscr{L}(0)=1$ and all the roots of $\mathscr{L}(u)$ are on the circle of radius $r^{-\frac{d}{2}}$ for some non-negative integer number d which is called the weight of the L-function. We say that the degree of the polynomial $\mathscr{L}(u)$ is the degree of the corresponding L-function. A zeta function $\zeta(s)$ is a product of L-functions in powers ±1:

$$\zeta(s)=\prod_{k=0}^{d} L_k(s)^{w_k},$$

where $w_k\in\{-1,1\}$ and $L_k(s)$ is an L-function of weight k.

Both zeta functions of smooth projective curves or even varieties over finite fields and L-functions of elliptic surfaces considered in the previous sections are covered by this definition.

For the logarithm of a zeta function we have the Dirichlet series expansion:

$$\log\zeta(s) = \sum_{f=1}^{\infty} \frac{\Lambda_f}{f} r^{-fs}$$

which is convergent for $\operatorname{Re} s > \frac{d}{2}$. In the case of a variety $X/\mathbb{F}_r$ we have a simple interpretation for the coefficients $\Lambda_f = |X(\mathbb{F}_{r^f})|$ as the number of points on X over the degree f extension of $\mathbb{F}_r$.

We are going to work with zeta and L-functions asymptotically, so we have to introduce the notion of a family. We will call a sequence $\{\zeta_k(s)\}_{k=1\ldots\infty} = \left\{\prod_{i=0}^{d} L_{ki}(s)^{w_i}\right\}_{k=1\ldots\infty}$ of zeta functions a family if the total degree $g_k = \sum_{i=0}^{d} g_{ki}$ tends to infinity and d remains constant. Here g_{ki} are the degrees of the individual L-functions $L_{ki}(s)$ in $\zeta_k(s)$.

Definition 5.8. A family $\{\zeta_k(s)\}_{k=1\ldots\infty}$ of zeta functions is called asymptotically exact if the limits

$$\gamma_i = \lim_{k\to\infty} \frac{g_{ki}}{g_k} \quad \text{and} \quad \lambda_f = \lim_{k\to\infty} \frac{\Lambda_{kf}}{g_k}$$

exist for each $i = 0, \ldots, d$ and each $f \in \mathbb{Z}$, $f \geqslant 1$. The family is called asymptotically bad if $\lambda_f = 0$ for any f and asymptotically good otherwise.

In the case of curves over finite fields the denominators of zeta functions are negligible from the asymptotic point of view. In general we give the following definition:

Definition 5.9. Let $\{\zeta_k(s)\}$ be an asymptotically exact family of zeta functions. Define the set $I \subset \{0\ldots d\}$ by the condition $i \in I$ if and only if $\gamma_i = 0$. We define $\zeta_{\mathbf{n},k}(s) = \prod_{i\in I} L_{ki}(s)^{w_i}$ the negligible part of $\zeta_k(s)$ and $\zeta_{\mathbf{e},k}(s) = \prod_{i\in\{0,\ldots,d\}\setminus I} L_{ki}(s)^{w_i}$ the essential part of $\zeta_k(s)$. Define also $d_{\mathbf{e}} = \max\{i \mid i \notin I\}$.

Definition 5.10. We say that an asymptotically exact family of zeta or L-functions is asymptotically very exact if the series

$$\sum_{f=1}^{\infty} |\lambda_f| q^{-\frac{fd_{\mathbf{e}}}{2}}$$

is convergent.

In the case of curves or varieties the positivity of Λ_f automatically implies the fact that the corresponding family is asymptotically very exact. This is of course false in general (an obvious example of a family which is asymptotically exact but not very exact is given by $L_k(s) = (1-q^{-s})^k$). In general most of the results are proven for asymptotically very exact families and not just for asymptotically exact ones.

We have already noted that the concept of limit zeta functions is of utmost importance in the asymptotic theory.

Definition 5.11. Let $\{\zeta_k(s)\}$ be an asymptotically exact family of zeta functions. Then the corresponding limit zeta function is defined as

$$\zeta_{\mathbf{lim}}(s) = \exp\biggl(\sum_{f=1}^{\infty} \frac{\lambda_f}{f} q^{-fs}\biggr).$$

Now, we can state the generalizations of most of the results concerning zeta and L-functions over finite fields, given in the previous sections. Let us begin with the basic inequalities. In fact, one should be able to write most of the inequalities from Subsection 5.1 in this more general setting. We give only the simplest statement of this type here:

Theorem 5.12. *Let $\{\zeta_k(s)\}$ be an asymptotically very exact family of zeta functions. Then*

$$w_{d_{\mathrm{e}}} \sum_{j=1}^{\infty} \lambda_j q^{-\frac{d_{\mathrm{e}} j}{2}} \leqslant \sum_{i=0}^{d_{\mathrm{e}}} \frac{\gamma_i}{q^{(d_{\mathrm{e}}-i)/2} + w_i}.$$

The Brauer—Siegel type results can also be proven in this setting. The following theorem includes all the function field versions of the Brauer—Siegel type results from Section 3 except for the explicit ones (which can also be, in principle, established for general zeta and L-functions).

Theorem 5.13. 1) *For any asymptotically exact family of zeta functions $\{\zeta_k(s)\}$ and any s with $\operatorname{Re} s > \frac{d_{\mathrm{e}}}{2}$ we have*

$$\lim_{k\to\infty} \frac{\log \zeta_{\mathrm{e},k}(s)}{g_k} = \log \zeta_{\mathbf{lim}}(s).$$

If, moreover, $2\operatorname{Re} s \notin \mathbb{Z}$, then

$$\lim_{k\to\infty} \frac{\log \zeta_k(s)}{g_k} = \log \zeta_{\mathbf{lim}}(s).$$

The convergence is uniform in any domain $\frac{d_{\mathrm{e}}}{2} + \varepsilon < \operatorname{Re} s < \frac{d_{\mathrm{e}}+1}{2} - \varepsilon$, $\varepsilon \in \left(0, \frac{1}{2}\right)$.

2) *If* $\{\zeta_k(s)\}$ *is an asymptotically very exact family with* $w_{d_{\mathbf{e}}}=1$ *we have*:

$$\lim_{k\to\infty}\frac{\log|c_k|}{g_k}\leqslant\log\zeta_{\mathbf{lim}}\Big(\frac{d_{\mathbf{e}}}{2}\Big),$$

where r_k *and* c_k *are defined using the Taylor series expansion*

$$\zeta_k(s)=c_k\Big(s-\frac{d_{\mathbf{e}}}{2}\Big)^{r_k}+O\Big(\Big(s-\frac{d_{\mathbf{e}}}{2}\Big)^{r_k+1}\Big).$$

In the case of arbitrary L-functions the equality in (2) does not hold in general. This means that the similar questions previously discussed for function fields or elliptic curves over function fields are indeed of arithmetic nature.

Finally we will state a result on the distribution of zeroes. Let $L(s)$ be an L-function and let $\rho_1,\ldots,\rho_g$ be the zeroes of the corresponding polynomial $\mathcal{L}(u)$. Define $\theta_k\in(-\pi,\pi]$ by $\rho_k=q^{-d/2}e^{i\theta_k}$. One can associate the measure $\Delta_L=\frac{1}{g}\sum\limits_{k=1}^{g}\delta_{\theta_k}$ to $L(s)$.

Theorem 5.14. *Let* $\{L_j(s)\}$ *be an asymptotically very exact family of L-functions. Then the limit distribution* $\lim\limits_{j\to\infty}\Delta_j$ *exists and has a nonnegative continuous density function given by an absolutely and uniformly convergent series* $1-2\sum\limits_{k=1}^{\infty}\lambda_k\cos(kx)q^{-\frac{dk}{2}}$.

In the case of families of elliptic curves over $\mathbb{F}_r(t)$ P. Michel provides in [27] an explicit estimate for the discrepancy in the equidistribution of zeroes and a much more precise estimate for it on average.

A number of open questions concerning asymptotic properties of zeta and L-functions can be found in the last section of [48]. It seems that an analogue of this general asymptotic theory can be developed in the number field case (at least assuming some plausible conjectures like GRH or the Ramanujan—Peterson conjecture). This is yet to be done.

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Asymptotic properties of zeta functions over finite fields

Abstract. In this paper we study asymptotic properties of families of zeta and L-functions over finite fields. We do it in the context of three main problems: the basic inequality, the Brauer—Siegel type results and the results on distribution of zeroes. We generalize to this abstract setting the results of Tsfasman, Vlăduţ and Lachaud, who studied similar problems for curves and (in some cases) for varieties over finite fields. In the classical case of zeta functions of curves we extend a result of Ihara on the limit behaviour of the Euler—Kronecker constant. Our results also apply to L-functions of elliptic surfaces over finite fields, where we approach the Brauer—Siegel type conjectures recently made by Kunyavskii, Tsfasman and Hindry.

1. Introduction

The origin of the asymptotic theory of global fields and their zeta functions can be traced back to the following classical question: what is the maximal number of points $N_q(g)$ on a smooth projective curve of genus g over the finite field $\mathbb{F}_q$. The question turns out to be difficult and a wide variety of methods has been used for finding both upper and lower bounds.

The classical bound of Weil stating that

$$|N_q(g) - q - 1| \leqslant 2g\sqrt{q}$$

though strong turns out to be far from optimal. A significant improvement for it when g is large was obtained by Drinfeld and Vlăduţ [3]. Namely, they proved that $\limsup\limits_{g\to\infty} \dfrac{N_q(g)}{g} \leqslant \sqrt{q} - 1$.

This inequality was a starting point for an in-depth study of asymptotic properties of curves over finite fields and of their zeta functions initiated by Tsfasman and Vlăduţ. This work went far beyond this initial inequality and has led to the introduction of the concept of limit zeta function which turned out to be very useful [21]. It also had numerous applications to coding theory via the so-called algebraic geometric codes (see, for example, the book [23] for some of them).

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The above mentioned study of limit zeta functions for families curves involves three main topics:

(1) The basic inequality, which is a generalization of the Drinfeld – Vlăduţ inequality on the number of points on curves.

(2) Brauer—Siegel type results which are the extensions of the classical Brauer—Siegel theorem describing the asymptotic behaviour of the class numbers and of the regulators in families of number fields. Here asymptotic properties of the special values of zeta functions of curves (such as the order of the Picard group) are studied.

(3) The distribution of zeroes of zeta functions (Frobenius eigenvalues) in families of curves.

There are at least two main directions in the further study of these topics. First, one may ask what are the number field counterparts of these results (for number fields and function fields are regarded by many as facets of a single gemstone). The translation of these results to the number field case is the subject of the paper [22]. The techniques turns out to be very analytically involved but the reward is no doubts significant as the authors managed to resolve some of the long standing problems (such as the generalization of the Brauer—Siegel theorem to the asymptotically good case, that is when the ratio $n_K/\log|D_K|$ of the degree to the logarithm of the discriminant does not tend to zero) as well as to improve several difficult results (Odlyzko—Serre inequalities for the discriminant, Zimmert's bound for regulators).

Second, one may ask what happens with higher dimensional varieties over finite fields. Here the answers are less complete. The first topic (main inequalities) was extensively studied in [12]. The results obtained there are fairly complete, though they do not directly apply to L-functions (such as L-functions of elliptic curves over function fields). The second topic is considerably less developed though it received some attention in the recent years in the case of elliptic surfaces [5], [11] and in the case of zeta functions of varieties over finite fields [24]. As for the results on the third topic one can cite a paper by Michel [14] where the case of elliptic curves over $\mathbb{F}_q(t)$ is treated. Quite a considerable attention was devoted to some finer questions related to the distribution of zeroes [10]. However, to our knowledge, not a single result of this type for asymptotically good families of varieties was previously known.

The goal of our paper is to study the above three topic in more generality separating fine arithmetic considerations from a rather simple (in the function field case) analytic part. We take the axiomatic approach, defining a class of L-functions to which our results may be applicable. This can be regarded as the function field analogue of working with the Selberg class in characteristic zero, though obviously the analytic contents in the function case is much less substantial (and often times even negligible). In our investigations we devote more attention to the second and the third topics (Brauer—Siegel type results and distribution of zeroes respectively) as being less developed then the first one. So, while giving results on the basic inequality, we do not seek to prove them in utmost generality (like in the paper [12]). We hope that this allows us to gain in clarity of the presentation as well as to save a considerable amount of space.

We use families of L-functions of elliptic curves over function fields as our motivating example. After each general statement concerning any of the three topics we specify what concrete results we get for zeta functions of curves, zeta functions of varieties over finite fields, and L-functions of elliptic curves over function fields. Our statements about the distribution of zeroes (Theorem 4.1 and Corollary 4.9) imply in the case of elliptic curves over function fields a generalization of a result due to Michel [14] (however, unlike us, Michel provides a rather difficult estimate for the error term). In the study of the Brauer—Siegel type results we actually manage to find something new even in the classical case of zeta functions of curves, namely we prove a statement on the limit behaviour of zeta functions of which the Brauer—Siegel theorem from [21] is a particular case (see Theorem 5.5 and Corollary 5.14). We also reprove and extend some of Ihara's results on Euler—Kronecker constant of function fields [6] incorporating them in the same general framework of limit zeta functions (see Corollary 5.16).

Here is the plan of our paper. In Section 2 we present the axiomatic framework for zeta and L-functions with which we will be working, then we give the so called explicit formulae for them. In the end of the section we introduce several particular examples coming from algebraic geometry (zeta functions of curves, zeta functions of varieties over finite fields, L-functions of elliptic curves over function fields) to which we will apply the general results. Each further section contains a subsection where we show what the results on abstract zeta and L-functions give in these concrete cases. In Section 3 we outline the asymptotic approach to the study of zeta and L-functions, introducing the notions of asymptotically exact and asymptotically very exact families. We prove the zero distribution results in Section 4. There we also give some applications to the distribution of zeroes and the growth of analytic ranks in families of elliptic surfaces (Corollaries 4.9 and 4.11). The study of the Brauer—Siegel type results is undertaken in Section 5. In the same section we show how these results imply the formulae for the asymptotic behaviour of the invariants of function fields generalizing the Euler—Kronecker constant (Corollary 5.16) and a certain bound towards the conjectures of Kunyavskii—Tsfasman and Hindry—Pacheko (Theorem 5.27). Section 6 is devoted to the proof of several versions of the basic inequality. In this section we generalize some of the results from [12] to the case of zeta and L-functions with not necessarily positive coefficients. Finally, in Section 7 we discuss some possible further development as well as open questions.

2. Zeta and L-functions

2.1. Definitions

Let us define the class of L-functions we will be working with. Let $\mathbb{F}_q$ be the finite field with q elements.

Definition 2.1. An L-function $L(s)$ over a finite field $\mathbb{F}_q$ is a holomorphic function in s such that for $u = q^{-s}$ the function $\mathscr{L}(u) = L(s)$ is a polynomial with real coefficients, $\mathscr{L}(0) = 1$ and all the roots of $\mathscr{L}(u)$ are on the circle of radius $q^{-\frac{w}{2}}$ for some non-negative integer number w.

We will refer to the last condition in the definition as the Riemann hypothesis for $L(s)$ since it is the finite field analogue of the classical Riemann hypothesis for the Riemann zeta function. The number w in the definition of an L-function will be called its *weight*. We will also say that the degree d of the polynomial $\mathscr{L}(u)$ is the *degree* of the L-function $L(s)$ (it should not be confused with the degree of an L-function in the analytic number theory, where it is taken to be the degree of the polynomial in its Euler product).

The logarithm of an L-function has a Dirichlet series expansion

$$\log L(s) = \sum_{f=1}^{\infty} \frac{\Lambda_f}{f} q^{-fs},$$

which converges for $\operatorname{Re} s > \dfrac{w}{2}$. For the opposite of the logarithmic derivative we get the formula:

$$-\frac{L'(s)}{L(s)} = \sum_{f=1}^{\infty} (\Lambda_f \log q)\, q^{-fs} = u\,\frac{\mathscr{L}'(u)}{\mathscr{L}(u)} \log q.$$

There is a functional equation for $L(s)$ of the form

$$L(w-s) = \omega q^{\left(s-\frac{w}{2}\right)d} L(s), \tag{2.1}$$

where $d = \deg \mathscr{L}(u)$ and $\omega = \pm 1$ is the root number. This can be proven directly as follows. Let $\mathscr{L}(u) = \prod_{i=1}^{d}\left(1 - \frac{u}{\rho_i}\right)$. Then

$$\mathscr{L}\left(\frac{1}{uq^w}\right) = \prod_{i=1}^{d}\left(1 - \frac{1}{\rho_i u q^w}\right) = \prod_{i=1}^{d} \rho_i \cdot q^{-wd} u^{-d} \prod_{i=1}^{d}\left(\frac{u}{\bar{\rho}_i} - 1\right) =$$

$$= (-1)^{d-t} q^{\frac{-wd}{2}} u^{-d} \prod_{i=1}^{d}\left(1 - \frac{u}{\rho_i}\right),$$

where t is the multiplicity of the root $-q^{w/2}$. We used the fact that all coefficients of $\mathscr{L}(u)$ are real, so its non-real roots come in pairs ρ and $\bar{\rho}$, $\rho\bar{\rho} = q^w$.

Definition 2.2. A zeta function $\zeta(s)$ over a finite field $\mathbb{F}_q$ is a product of L-functions in powers ± 1:

$$\zeta(s) = \prod_{i=0}^{w} L_i(s)^{\varepsilon_i},$$

where $\varepsilon_i \in \{-1, 1\}$, $L_i(s)$ is an L-function of weight i.

For the logarithm of a zeta function we also have the Dirichlet series expansion:

$$\log \zeta(s) = \sum_{f=1}^{\infty} \frac{\Lambda_f}{f} q^{-fs}$$

which is convergent for $\operatorname{Re} s > \frac{w}{2}$.

2.2. Explicit formulae

In this subsection we will derive the analogues of Weil and Stark explicit formulae for our zeta and L-functions. The proofs of the Weil explicit formula can be found in [15] for curves and in [12] for varieties over finite fields. An explicit formula for L-functions of elliptic surfaces is proven in [1]. In our proof we will follow the latter exposition.

Recall that our main object of study is $\zeta(s) = \prod_{i=0}^{w} L_i(s)^{\varepsilon_i}$ a zeta function with $L_i(s)$ given by

$$L_i(s) = \prod_{j=1}^{d_i}\left(1 - \frac{q^{-s}}{\rho_{ij}}\right).$$

As before, we define Λ_f via the relation $\log \zeta(s) = \sum_{f=1}^{\infty} \frac{\Lambda_f}{f} q^{-fs}$.

Proposition 2.3. *Let* $\mathbf{v}=(v_f)_{f\geqslant 1}$ *be a sequence of real numbers and let* $\psi_{\mathbf{v}}(t)=\sum_{f=1}^{\infty} v_f t^f$. *Let* $\rho_{\mathbf{v}}$ *be the radius of convergence of the series* $\psi_{\mathbf{v}}(t)$. *Assume that* $|t|<q^{-w/2}\rho_v$, *then*

$$\sum_{f=1}^{\infty} \Lambda_f v_f t^f = -\sum_{i=0}^{w} \varepsilon_i \sum_{j=1}^{d_i} \psi_{\mathbf{v}}(q^i \rho_{ij} t).$$

Proof. Let us prove this formula for L-functions. The formula for zeta functions will follow by additivity.

The simplest is to work with $\mathscr{L}(u)=\prod_{j=1}^{d}\left(1-\frac{u}{\rho}\right)$. The coefficient of u^f in $-u\mathscr{L}'(u)/\mathscr{L}(u)$ is seen to be $\sum_{\rho}\rho^{-f}$ for $f\geqslant 1$. From this we derive the equality:

$$\sum_{\rho}\rho^{-f} = -\Lambda_f.$$

The map $\rho\mapsto(q^w\rho)^{-1}$ permutes the zeroes $\{\rho\}$, thus for any $f\geqslant 1$ we have:

$$\sum_{\rho}(q^w\rho)^f = -\Lambda_f.$$

Multiplying the last identity by $v_f t^f$ and summing for $f=1,2,\ldots$ we get the statement of the theorem. □

From this theorem one can easily get a more familiar version of the explicit formula (like the one from [15] in the case of curves over finite fields).

Corollary 2.4. *Let* $L(s)$ *be an* L*-function, with zeroes* $\rho=q^{-w/2}e^{i\theta}$, $\theta\in[-\pi,\pi]$. *Let* $f:[-\pi,\pi]\to\mathbb{C}$ *be an even trigonometric polynomial*

$$f(\theta)=v_0+2\sum_{n=1}^{Y} v_n\cos(n\theta).$$

Then we have the explicit formula:

$$\sum_{\theta} f(\theta) = v_0 d - 2\sum_{f=1}^{Y} v_f\Lambda_f q^{-\frac{wf}{2}}.$$

Proof. We put $t=q^{-\frac{w}{2}}$ in the above explicit formula and notice that the sum over zeroes can be written using cos since all the non-real zeroes come in complex conjugate pairs. □

In the next sections we will also make use of the so called Stark formula (which borrows its name from its number field counterpart from [18]).

Proposition 2.5. *For a zeta function $\zeta(s)$ we have*:

$$\frac{1}{\log q}\frac{\zeta'(s)}{\zeta(s)} = \sum_{i=0}^{w}\varepsilon_i\sum_{j=1}^{d_i}\frac{1}{q^s\rho_{ij}-1} = -\frac{1}{2}\sum_{i=0}^{w}\varepsilon_i d_i + \frac{1}{\log q}\sum_{i=0}^{w}\varepsilon_i\sum_{L_i(\theta_{ij})=0}\frac{1}{s-\theta_{ij}},$$

we assume that $q^{-\theta_{ij}}=\rho_{ij}$, the sum is taken over all possible roots θ_{ij} counted with multiplicity.

Proof. The first equality is a trivial consequence of the formulae expressing $\mathscr{L}_i(u)$ as polynomials in u.

The second equality follows from the following series expansion:

$$\frac{\log q}{\rho^{-1}q^s-1}+\frac{\log q}{2} = \lim_{T\to\infty}\sum_{\substack{q^\theta=\rho\\|\theta|\leqslant T}}\frac{1}{s-\theta}. \qquad \square$$

2.3. Examples

We have in mind three main types of examples: zeta functions of curves over finite fields, zeta functions of varieties over finite fields and L-functions of elliptic curves over function fields.

Example 2.6 (Curves over finite fields)**.** Let X be an absolutely irreducible smooth projective curve of genus g over the finite field $\mathbb{F}_q$ with q elements. Let Φ_f be the number of points of degree f on X. The zeta function of X is defined for $\operatorname{Re} s>1$ as

$$\zeta_X(s) = \prod_{f=1}^{\infty}(1-q^{-fs})^{-\Phi_f}.$$

It is known that $\zeta_X(s)$ is a rational function in $u=q^{-s}$. Moreover,

$$\zeta_K(s) = \frac{\prod_{j=1}^{g}\Bigl(1-\frac{u}{\rho_j}\Bigr)\Bigl(1-\frac{u}{\bar\rho_j}\Bigr)}{(1-u)(1-qu)},$$

and $|\rho_j|=q^{-\frac{1}{2}}$ (Weil's theorems). Note that the functional equation implies that the real Frobenius roots all have even multiplicity. It can easily be seen that in this case $\Lambda_f=N_f(X)$ is the number of points on $X\otimes_{\mathbb{F}_q}\mathbb{F}_{q^f}$ over $\mathbb{F}_{q^f}$. A very important feature of this example which will be lacking in general is that $\Lambda_f\geqslant 0$ for all f.

Though $\zeta_X(s)$ is not an L-function, in all asymptotic considerations the denominator will be irrelevant and it will behave as an L-function.

This example will serve as a motivation in most of our subsequent considerations, for most (but not all, see Section 5) of the results we derive for general zeta and L-functions are known in this setting.

Example 2.7 (Varieties over finite fields)**.** Let X be a non-singular absolutely irreducible projective variety of dimension n defined over a finite field $\mathbb{F}_q$. Denote by $|X|$ the set of closed points of X. We put $X_f = X \otimes_{\mathbb{F}_q} \mathbb{F}_{q^f}$ and $\bar{X} = X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$. Let Φ_f be the number of points of X having degree f, that is $\Phi_f = |\{v \in |X| \mid \deg(v) = f\}|$. The number N_f of $\mathbb{F}_{q^f}$-points of the variety X_f is equal to $N_f = \sum\limits_{m|f} m\Phi_m$.

Let $b_s(X) = \dim_{\mathbb{Q}_l} H^s(\bar{X}, \mathbb{Q}_l)$ be the l-adic Betti numbers of X. The zeta function of X is defined for $\mathrm{Re}(s) > n$ by the following Euler product:

$$\zeta_X(s) = \prod_{v \in |X|} \frac{1}{1 - (\mathrm{N}v)^{-s}} = \prod_{f=1}^{\infty} (1 - q^{-fs})^{-\Phi_f},$$

where $\mathrm{N}v = q^{-\deg v}$. If we set $Z_X(u) = \zeta_X(s)$ with $u = q^{-s}$ then the function $Z_X(u)$ is a rational function of u and can be expressed as

$$Z_X(u) = \prod_{i=0}^{2n} P_i(X, u)^{(-1)^{i-1}},$$

where

$$P_i(X, u) = \prod_{j=1}^{d_i} \Big(1 - \frac{u}{\rho_{ij}}\Big),$$

and $|\rho_{ij}| = q^{-i/2}$ (Weil's conjectures proven by Deligne). Moreover,

$$P_0(X, u) = 1 - u \quad \text{and} \quad P_{2n}(X, u) = 1 - q^n u.$$

As before, we have that $\Lambda_f = N_f(X) \geqslant 0$.

The previous example is obviously included in this one. However, it is better to separate them as in the case of zeta functions of general varieties over finite fields much less is known. One more reason to distinguish between these two examples is that, whereas zeta functions of curves asymptotically behave as L-functions, zeta functions of varieties are “real” zeta functions. Thus there is quite a number of properties that simply do not hold in general (for example, some of those connected to the distribution of zeroes).

Example 2.8 (Elliptic curves over function fields)**.** Let E be a non-constant elliptic curve over a function field $K = \mathbb{F}_q(X)$ with finite constant field $\mathbb{F}_q$. The curve E can also be regarded as an elliptic surface over $\mathbb{F}_q$. Let g be the genus of X. Places of K (that is points of X) will be denoted by v. Let $d_v = \deg v$, $|v| = \mathrm{N}v = q^{\deg v}$ and let $\mathbb{F}_v = \mathbb{F}_{\mathrm{N}v}$ be the residue field of v.

For each place v of K we define a_v from $|E_v(\mathbb{F}_v)| = |v| + 1 - a_v$, where $|E_v(\mathbb{F}_v)|$ is the number of points on the reduction E_v of the curve E. The local factors $L_v(s)$ are defined by

$$L_v(s) = \begin{cases} (1 - a_v|v|^{-s} + |v|^{1-2s})^{-1}, & \text{if } E_v \text{ is non-singular;} \\ (1 - a_v|v|^{-s})^{-1}, & \text{otherwise.} \end{cases}$$

We define the global L-function $L_E(s) = \prod_v L_v(s)$. The product converges for $\operatorname{Re} s > \frac{3}{2}$ and defines an analytic function in this half-plane. Define the conductor N_E of E as the divisor $\sum_v n_v v$ with $n_v = 1$ at places of multiplicative reduction, $n_v = 2$ at places of additive reduction for $\operatorname{char} \mathbb{F}_q > 3$ (and possibly larger when $\operatorname{char} \mathbb{F}_q = 2$ or 3) and $n_v = 0$ otherwise. Let $n_E = \deg N_E = \sum_v n_v \deg v$.

It is known (see [1]) that $L_E(s)$ is a polynomial $\mathscr{L}_E(u)$ in $u = q^{-s}$ of degree $n_E + 4g - 4$. The polynomial $\mathscr{L}_E(u)$ has real coefficients, satisfies $\mathscr{L}_E(0) = 1$ and all of its roots have absolute value q^{-1}.

Let $\alpha_v, \bar{\alpha}_v$ be the roots of the polynomial $1 - a_v t + |v| t^2$ for a place v of good reduction and let $\alpha_v = a_v$ and $\bar{\alpha}_v = 0$ for a place v of bad reduction. Then from the definition of $L_E(s)$ one can easily deduce that

$$\Lambda_f = \sum_{m d_v = f} d_v (\alpha_v^m + \bar{\alpha}_v^m), \tag{2.2}$$

the sum being taken over all places v of K and $m \geqslant 1$ such that $m \deg v = f$.

This example will be the principal one in the sense that all our results on L-functions are established in the view to apply them to this particular case. These L-functions are particularly interesting from the arithmetic point of view, especially due to the connection between the special value of such an L-function at $s = 1$ and the arithmetic invariants of the elliptic curve (the order of the Shafarevich—Tate group and the regulator) provided by the Birch and Swinnerton-Dyer conjecture.

We could have treated the more general example of abelian varieties over function fields. However, we prefer to restrict ourselves to the case elliptic curves to avoid technical complications.

3. Families of zeta and L-functions

3.1. Definitions and basic properties

We are interested in studying sequences of zeta and L-functions. Let us fix the finite field $\mathbb{F}_q$.

Definition 3.1. We will call a sequence $\{L_k(s)\}_{k=1\ldots\infty}$ of L-functions a family if they all have the same weight w and the degree d_k tends to infinity.

Definition 3.2. We will call a sequence

$$\{\zeta_k(s)\}_{k=1\ldots\infty} = \left\{\prod_{i=0}^{w} L_{ik}(s)^{\varepsilon_i}\right\}_{k=1\ldots\infty}$$

of zeta functions a family if the total degree $\widetilde{d}_k = \sum\limits_{i=0}^{w} d_{ik}$ tends to infinity. Here d_{ik} are the degrees of the individual L-functions $L_{ik}(s)$ in $\zeta_k(s)$.

Remark 3.3. In the definition of a family of zeta functions we assume that $w = w_k$ and $\varepsilon_i = \varepsilon_{ik}$ are the same for all k. Obviously, any family of L-functions is at the same time a family of zeta functions.

Definition 3.4. A family $\{\zeta_k(s)\}_{k=1\ldots\infty}$ of zeta or L-functions is called asymptotically exact if the limits

$$\delta_i = \delta_i(\{\zeta_k(s)\}) = \lim_{k\to\infty} \frac{d_{ik}}{\widetilde{d}_k} \quad\text{and}\quad \lambda_f = \lambda_f(\{\zeta_k(s)\}) = \lim_{k\to\infty} \frac{\Lambda_{fk}}{\widetilde{d}_k}$$

exist for each $i = 0, \ldots, w$ and each $f \in \mathbb{Z}$, $f \geqslant 1$. It is called asymptotically bad if $\lambda_f = 0$ for any f and asymptotically good otherwise.

The following (easy) proposition will be important.

Proposition 3.5. *Let $L(s)$ be an L-function. Then*

1) *for each f we have the bound $|\Lambda_f| \leqslant q^{\frac{wf}{2}} d$;*

2) *there exists a number $C(q, w, s)$ depending on q, w and s but not on d such that $|\log L(s)| \leqslant C(q, w, s)d$ for any s with $\operatorname{Re} s \neq \frac{w}{2}$. The number $C(q, w, s)$ can be chosen independent of s if s belongs to a vertical strip $a \leqslant \operatorname{Re} s \leqslant b$, $\frac{w}{2} \notin [a, b]$.*

Proof. To prove the first part we use Proposition 2.1. Applying it to the sequence consisting of one non-zero term we get:

$$\Lambda_f = -\sum_{\mathscr{L}(\rho)=0} q^{wf} \rho^f. \tag{3.1}$$

The absolute value of the right hand side of this equality is bounded by $q^{\frac{wf}{2}} d$.

To prove the second part we assume first that $\operatorname{Re} s = \varepsilon + \frac{w}{2} > \frac{w}{2}$. We have the estimate:

$$|\log L(s)| = \left| \sum_{f=1}^{\infty} \frac{\Lambda_f}{f} q^{-fs} \right| \leqslant \sum_{f=1}^{\infty} \frac{d}{f} \cdot q^{\frac{wf}{2}} \cdot q^{-f \operatorname{Re} s} \leqslant d \sum_{f=1}^{\infty} \frac{1}{f q^{\varepsilon f}}.$$

For $\operatorname{Re} s < \frac{w}{2}$ we use the functional equation (2.1). □

Proposition 3.6. *Any family of zeta and L-functions contains an asymptotically exact subfamily.*

Proof. We note that both $\frac{d_{ik}}{\widetilde{d}_k}$ and $\frac{\Lambda_{fk}}{\widetilde{d}_k}$ are bounded. For the first expression it is obvious and the second expression is bounded by Proposition 3.5. Now we can use the diagonal method to choose a subfamily for which all the limits exist. □

As in the case of curves over finite fields we have to single out the factors in zeta functions which are asymptotically negligible. This can be done using Proposition 3.5.

Definition 3.7. Let $\{\zeta_k(s)\}$ be an asymptotically exact family of zeta functions. Define the set $I \subset \{0, \ldots, w\}$ by the condition $i \in I$ if and only if $\delta_i = 0$. We define $\zeta_{\mathbf{n},k}(s) = \prod_{i \in I} L_{ik}(s)^{\varepsilon_i}$ the negligible part of $\zeta_k(s)$ and $\zeta_{\mathbf{e},k}(s) = \prod_{i \in \{0,\ldots,w\} \setminus I} L_{ik}(s)^{\varepsilon_i}$ the essential part of $\zeta_k(s)$. Define also $w_{\mathbf{e}} = \max\{i \in \{0, \ldots, w\} \setminus I\}$.

Remark 3.8. The functions $\zeta_{\mathbf{n},k}(s)$ and $\zeta_{\mathbf{e},k}(s)$ make sense only for families of zeta functions and not for individual zetas. We also note that the definitions of the essential and the negligible parts are obviously trivial for families of L-functions.

The following proposition, though rather trivial, turns out to be useful.

Proposition 3.9. *For an asymptotically exact family of zeta functions* $\{\zeta_k(s)\}$ *we have* $\lambda_f(\zeta_k(s)) = \lambda_f(\zeta_{\mathbf{e},k}(s))$.

Proof. This is an immediate corollary of Proposition 3.5. □

The condition on a family to be asymptotically exact suffices for applications in the case of varieties over finite fields due to the positivity of coefficients Λ_f. However, in general we will have to impose somewhat more restrictive conditions on the families.

Definition 3.10. We say that an asymptotically exact family of zeta or L-functions is asymptotically very exact if the series

$$\sum_{f=1}^{\infty} |\lambda_f| q^{-\frac{f w_{\mathbf{e}}}{2}}$$

is convergent.

Example 3.11. An obvious example of a family which is asymptotically exact but not very exact is given by the family of L-functions $L_k(s)=(1-q^{-s})^k$. We have $\lambda_f=-1$ for any f and the series $\sum\limits_{f=1}^{\infty}(-1)$ is clearly divergent.

Proposition 3.12. *Assume that we have an asymptotically exact family of zeta functions*

$$\{\zeta_k(s)\}=\left\{\prod_{i=0}^{w} L_{ik}(s)^{\varepsilon_i}\right\}_{k=1\dots\infty},$$

such that all the families $\{L_{ik}(s)\}$ are also asymptotically exact. Then, the family $\{\zeta_k(s)\}$ is asymptotically very exact if and only if the family $\{L_{w_{\mathrm{e}}k}(s)\}$ is asymptotically very exact.

Proof. This follows from Proposition 3.5 together with Proposition 3.9. □

In practice, this proposition means that the asymptotic behaviour of zeta functions for $\operatorname{Re} s>\dfrac{w_{\mathrm{e}}-1}{2}$ is essentially the same as that of their weight w_{e} parts. Thus, most asymptotic questions about zeta functions are reduced to the corresponding question about L-function.

3.2. Examples

As before we stick to three types of examples: curves over finite fields, varieties over finite fields and elliptic curves over function fields.

Example 3.13 (Curves over finite fields)**.** Let $\{X_j\}$ be a family of curves over $\mathbb{F}_q$. Recall (see [21]) that an asymptotically exact family of curves was defined by Tsfasman and Vlăduţ as such that the limits

$$\phi_f=\lim_{j\to\infty}\frac{\Phi_f(X_j)}{g_j} \tag{3.2}$$

exist. This is equivalent to our definition since $\Lambda_f=N_f(X)=\sum\limits_{m|f} m\Phi_m$. Note a little difference in the normalization of coefficients: in the case of curves we let $\lambda_f(\{X_j\})=\lim\limits_{j\to\infty}\dfrac{\Lambda_{fj}}{2g_j}$ since $2g_j$ is the degree of the corresponding polynomial in the numerator of $\zeta_{X_j}(s)$ and the authors of [21] choose to consider simply $\lim\limits_{j\to\infty}\dfrac{\Lambda_{fj}}{g_j}$.

For any asymptotically exact family of zeta functions of curves the negligible part of $\zeta_X(s)$ is its denominator $(1-q^{-s})(1-q^{1-s})$ and the essential part is its numerator. Thus, zeta functions of curves asymptotically behave like L-functions. Any asymptotically exact family of curves is asymptotically very exact as shows the basic inequality from [21] (see also Corollary 6.2 below), which is in fact due to positivity of Λ_f.

Example 3.14 (Varieties over finite fields). In the case of varieties of fixed dimension n over a finite field $\mathbb{F}_q$ we have an analogous notion of an asymptotically exact family [12], namely we ask for the existence of the limits

$$\phi_f = \lim_{j\to\infty} \frac{\Phi_f(X_j)}{b(X_j)} \quad \text{and} \quad \delta_i = \beta_i = \lim_{j\to\infty} \frac{b_i(X_j)}{b(X_j)},$$

where $b(X_j) = \sum_{i=0}^{2n} b_i(X_j)$ is the sum of Betti numbers. Again this definition and our Definition 3.4 are equivalent.

In this case the factors $(1-q^{-s})$ and $(1-q^{n-s})$ of the denominator are also always negligible. However, we can have more negligible factors as the following example shows.

Take the product $C\times C$, where C is a curve of genus $g\to\infty$. The dimension of the middle cohomology group H^2 grows as g^2 and $b_1 = b_3 = 2g$ (Kunneth formula). Thus $\zeta_{C\times C}(s)$ behaves like the inverse of an L-function.

If for an asymptotically exact family of varieties we have $w_{\mathrm{e}} = w-1 = 2n-1$ then it is asymptotically very exact as shows a form of the basic inequality [12, (8.8)] (it actually gives that the series $\sum_{f=1}^{\infty} \lambda_f q^{-f(n-1/2)}$ always converges), see also Corollary 6.7 below.

Example 3.15 (Elliptic curves over function fields). In the last example we will be interested in two particular types of asymptotically exact families.

Asymptotically bad families. Let us fix a function field $K = \mathbb{F}_q(X)$ and let us take the sequence of all pairwise non-isomorphic elliptic curves E_i/K. We get a family of L-functions since $n_{E_i}\to\infty$. From (2.2) we deduce that $|\Lambda_f| \leqslant 2\Big(\sum_{d_v|f} d_v\Big) q^{\frac{f}{2}}$ which is independent of n_{E_i}. Thus, this family is asymptotically exact and asymptotically bad, i. e. $\lambda_f = 0$ for any $f\geqslant 1$. This will be the only fact important for our asymptotic considerations. There will be no difference in the treatment of this particular family or in that of any other asymptotically bad family of L-functions.

This family is considered in [5] in the connection with the generalized Brauer—Siegel theorem. The main result of that paper is the reduction of the statement about the behaviour of the order of the Tate—

Shafarevich group and the regulator of elliptic curves over function fields to a statement about the values of their L-functions at $s=1$. See also [4] for a similar problem treated in the number field case.

Base change. Let us consider a family which is, in a sense, orthogonal to the previous one. Let $K=\mathbb{F}_q(X)$ be a function field and let E/K be an elliptic curve. Let $f\colon \mathscr{E}\to X$ be the corresponding elliptic surface. Consider a family of coverings of curves $X=X_0\leftarrow X_1\ldots\leftarrow X_i\leftarrow\ldots$ and the family of elliptic surfaces $\mathscr{E}_i$, given by the base change:

$$\begin{tikzcd}
\mathscr{E}=\mathscr{E}_0 \arrow[d, "f"] & \mathscr{E}_1 \arrow[l] \arrow[d] & \ldots \arrow[l] & \mathscr{E}_i \arrow[l] \arrow[d] & \ldots \arrow[l] \\
X=X_0 & X_1 \arrow[l] & \ldots \arrow[l] & X_i \arrow[l] & \ldots \arrow[l]
\end{tikzcd}$$

Let $\Phi_{v,f}(X_i)$ be the number of points on X_i, lying above a closed point $v\in|X|$, such that their residue fields have degree f over $\mathbb{F}_v$.

Lemma 3.16. *The limits*

$$\phi_{v,f}=\phi_{v,f}(\{X_i\})=\lim_{i\to\infty}\frac{\Phi_{v,f}(X_i)}{g(X_i)}$$

always exist.

Proof. We will follow the proof of the similar statement for Φ_f from [22, Lemma 2.4]. Let $K_2\supseteq K_1\supseteq K$ be finite extension of function fields. From the Riemann—Hurwitz formula we deduce the inequality $g(K_2)-1\geqslant[K_2\colon K_1](g(K_1)-1)$, where $[K_2:K_1]$ is the degree of the corresponding extension. Now, if we fix w a place of K_1 above v and consider its decomposition $\{w_1,\ldots,w_r\}$ in K_2, then we have

$$\sum_{i=1}^{r}\deg w_i\leqslant[K_2\colon K_1].$$

Thus we get for any $n\geqslant 1$ the inequality

$$\sum_{f=1}^{n}f\Phi_{v,f}(K_2)\leqslant[K_2\colon K_1]\sum_{f=1}^{n}f\Phi_{v,f}(K_1).$$

Dividing we see that

$$\frac{\sum\limits_{f=1}^{n}f\Phi_{v,f}(K_2)}{g(K_2)-1}\leqslant\frac{\sum\limits_{f=1}^{n}f\Phi_{v,f}(K_1)}{g(K_1)-1}.$$

It follows that the sequence $\sum_{f=1}^{n} \frac{f\Phi_{v,f}(X_i)}{g(X_i)-1}$ is non-increasing and non-negative for any fixed n and so has a limit. Taking $n=1$ we see that $\phi_{v,1}$ exists, taking $n=2$ we derive the existence of $\phi_{v,2}$ and so on. □

Let us remark that $\Phi_f(X_i) = \sum_{m \deg v = f} \Phi_{v,m}(X_i)$, the sum being taken over all places v of K and the same equality holds for ϕ_f (in particular, the family $\{X_i\}$ is asymptotically exact).

For our family we can derive a concrete expression for the Dirichlet series coefficients of the logarithms of L-functions. Indeed, (2.2) gives us

$$\Lambda_f = \sum_{mkd_v = f} md_v \Phi_{v,m}(\alpha_v^{mk} + \bar{\alpha}_v^{mk}). \tag{3.3}$$

Lemma 3.17. *Let E_i/K_i be a family of elliptic curves obtained by a base change and let $n_i = n_{E_i/K_i}$ be the degree of the conductor of E_i/K_i. Then the ratio $\frac{n_i}{g_i}$ is bounded by a constant depending only on E_0/K_0.*

If, furthermore, $\operatorname{char}\mathbb{F}_q \neq 2, 3$ *or the extensions K_i/K_0 are Galois for all i then the limit $\nu = \lim_{i\to\infty} \frac{n_i}{g_i}$ exists.*

Proof. The proof basically consists of looking at the definition of the conductor and applying the same method as in the proof of Lemma 3.16. Recall, that if E/K is an elliptic curve over a local field K, $T_l(E)$ is its Tate module, $l \neq \operatorname{char}\mathbb{F}_q$, $V_l(E) = T_l(E) \otimes \mathbb{Q}_l$, $I(\bar{K}/K)$ is the inertia subgroup of $\mathrm{Gal}(\bar{K}/K)$, then the tame part of the conductor is defined as

$$\varepsilon(E/K) = \dim_{\mathbb{Q}_l}(V_l(E)/V_l(E)^{I(\bar{K}/K)}).$$

It is easily seen to be non increasing in extensions of K, moreover it is known that $0 \leqslant \varepsilon(E/K) \leqslant 2$ (see [17, Chap. IV, § 10]).

If we let $L = K(E[l])$, $\gamma_i(L/K) = |G_i(L/K)|$, where $G_i(L/K)$ is the i^{th} ramification group of L/K, then the wild part of the conductor is defined as

$$\delta(E/K) = \sum_{i=1}^{\infty} \frac{\gamma_i(L/K)}{\gamma_0(L/K)} \dim_{\mathbb{F}_l}(E[l]/E[l]^{G_i(L/K)}).$$

One can prove [17, Chap. IV, § 10] that $\delta(E/K)$ is zero unless the characteristic of the residue field of K is equal to 2 or 3. In any case, the definition shows that $\delta(E/M)$ can take only finitely many values if we fix E and let vary the extension M/K.

The exponent of the conductor of E over the local field K is defined to be $f(E/K) = \varepsilon(E/K) + \delta(E/K)$. For an elliptic curve E over a global field

K the v-exponent of the conductor is taken to be $n_v(E/K) = f(E/K_v)$, where K_v is the completion of K at v.

From the previous discussion we see that for each valuation v of K_0 and each place w of K_i over v there is a constant c_v (depending on v and on K) such that $n_w(E/K_i) \leqslant c_v$. Thus

$$n_i = \sum_{w \in \mathrm{Val}(K_i)} n_w \deg w \leqslant \sum_{v \in \mathrm{Val}(K)} c_v \sum_{w|v} \deg w \leqslant \Big(\sum_{v \in \mathrm{Val}(K)} c_v \Big) \cdot [K_i : K_0],$$

so the ratio $\dfrac{n_i}{g_i}$ is bounded. If, furthermore, $\operatorname{char} \mathbb{F}_q \neq 2, 3$, then an argument similar to the one used in the proof of Lemma 3.16 together with the fact that $n_w(E) \leqslant n_v(E)$ if w lies above v in an extension of fields gives us that the sequence $\dfrac{n_i}{g_i}$ is non-increasing and so it has a limit $\nu = \nu(\{E_i/K_i\})$.

In the case of Galois extensions we notice that $n_w(E)$ must stabilize in a tower and all the $n_w(E)$ are equal for w over a fixed place v. Thus the previous argument is applicable once again. □

Now we can prove the following important proposition:

Proposition 3.18. *Any family of elliptic curves obtained by a base change contains an asymptotically very exact subfamily. If, furthermore,* $\operatorname{char} \mathbb{F}_q \neq 2, 3$ *or the extensions* K_i/K_0 *are Galois for all* i *then it is itself asymptotically very exact.*

Proof. Recall that for each E_i/K_i the degree of the corresponding L-function is $n_i + 4g_i - 4$. It follows from the previous lemma that it is enough to prove the existence of the limits $\widetilde{\lambda}_f = \lim\limits_{i\to\infty} \dfrac{\Lambda_f(E_i)}{g_i}$ and the convergence of the series $\sum\limits_{f=1}^{\infty} |\widetilde{\lambda}_f| q^{-f}$.

The first statement is a direct corollary of Lemma 3.16 and (3.3). As for the second statement, we have the following bound:

$$|\Lambda_f| \leqslant 2 \sum_{mkd_v = f} m d_v \Phi_{v,m} q^{\frac{f}{2}} = 2 \sum_{lk=f} l \Phi_l q^{\frac{f}{2}} = 2 N_f q^{\frac{f}{2}}.$$

Now, the convergence of the series $\sum\limits_{f=1}^{\infty} \nu_f q^{-\frac{f}{2}}$ with $\nu_f = \lim\limits_{i\to\infty} \dfrac{N_f(X_i)}{g_i}$ is a consequence of the basic inequality for zeta functions of curves ([20, Corollary 1] or Example 6.10). □

Remark 3.19. It would be nice to know whether the statement of the previous proposition holds without any additional assumptions, i. e.

whether a family obtained by a base change is always asymptotically very exact. This depends on Lemma 3.17, which we do not know how to prove in general.

The family of elliptic curves obtained by the base change was studied in [11] again in the attempts to obtain a generalization of the Brauer—Siegel theorem to this case. Kunyavskii and Tsfasman formulate a conjecture on the asymptotic behaviour of the order of the Tate—Shafarevich group and the regulator in such families (see Conjecture 5.25 below). They also treat the case of constant elliptic curves in more detail. Unfortunately, the proof of the main theorem [11, Theorem 2.1] given there is not absolutely flawless (the change of limits remains to be justified, which seems to be very difficult if not inaccessible at present).

Remark 3.20. If, for a moment, we turn our attention to general families of elliptic surfaces the following natural question arises:

Question 3.21. Is it true that any family of elliptic surfaces contains an asymptotically very exact subfamily?

The fact that it is true for two "orthogonal" cases makes us believe that this property might hold in general.

4. Distribution of zeroes

4.1. Main results

In this section we will prove certain results about the limit distribution of zeroes in families of L-functions. As a corollary we will see that the multiplicities of zeroes in asymptotically very exact families of L-functions can not grow too fast.

Let $C = C[0, \pi]$ be the space of real continuous functions on $[0, \pi]$ with topology of uniform convergence. The space of measures μ on $[0, \pi]$ is by definition the space $\mathcal{M}$, which is topologically dual to C. The topology on $\mathcal{M}$ is the $*$-weak one: $\mu_i \to \mu$ if and only if $\mu_i(f) \to \mu(f)$ for any $f \in C$.

The space C can be considered as a subspace of $\mathcal{M}$: if $\phi(x) \in C$ then

$$\mu_\phi(f) = \int_{-\pi}^{\pi} f(x)\phi(x)\, dx.$$

The subspace C is dense in $\mathcal{M}$ in $*$-weak topology.

Let $L(s)$ be an L-function and let $\rho_1, \ldots, \rho_d$ be the zeroes of the corresponding polynomial $\mathcal{L}(u)$. Define $\theta_k \in (-\pi, \pi]$ by $\rho_k = q^{-w/2} e^{i\theta_k}$. For a zero $\rho\in\{\rho_1,\dots,\rho_d\}$ we let

$$m_\rho=\begin{cases}\text{the multiplicity of }\rho, & \text{if } \rho\notin\mathbb{R};\\ \frac{1}{2}\cdot(\text{multiplicity of }\rho), & \text{if } \rho\in\mathbb{R} \text{ (that is } \rho=q^{-w/2}).\end{cases}$$

Since $\mathscr{L}(u)\in\mathbb{R}[u]$, we note that $m_{\bar\rho}=m_\rho$ for any zero ρ. We associate a measure to $L(s)$ in the following way

$$\mu_L(f)=\frac{2}{d}\sum_{\substack{\theta_k\geqslant 0\\ 1\leqslant k\leqslant d}} m_{\rho_k}\delta_{\theta_k}(f), \tag{4.1}$$

where δ_{θ_k} is the Dirac measure supported at θ_k, i. e. $\delta_{\theta_k}(f)=f(\theta_k)$ for an $f\in C$.

The main result of this section is the following one:

Theorem 4.1. *Let $\{L_j(s)\}$ be an asymptotically very exact family of L-functions. Then the limit $\mu_{\mathbf{lim}}=\lim\limits_{j\to\infty}\mu_{L_j}$ exists. Moreover, $\mu_{\mathbf{lim}}$ is a non-negative continuous function given by an absolutely and uniformly convergent series*:

$$\mu_{\mathbf{lim}}(x)=1-2\sum_{k=1}^{\infty}\lambda_k\cos(kx)q^{-\frac{wk}{2}}.$$

Proof. The absolute and uniform convergence of the series follows from the definition of an asymptotically very exact family. It is sufficient to prove the convergence of measures on the space $C[0,\pi]$.

Finite linear combinations of $\cos(nx)$ for $n\in\mathbb{N}$ are dense in the space of continuous functions $C[0,\pi]$, so it is enough to prove that for any $n=0,1,2,\dots$ we have:

$$\lim_{j\to\infty}\mu_{L_j}(\cos(nx))=\mu_{\mathbf{lim}}(\cos(nx)). \tag{4.2}$$

The Corollary 2.4 shows that:

$$\mu_{L_j}(\cos(nx))=\frac{2}{d_j}\sum_{\substack{\theta_{kj}\geqslant 0\\ 1\leqslant k\leqslant d_j}} m_{\rho_{kj}}\cos(n\theta_{kj})=\frac{1}{d_j}\sum_{k=1}^{d_j}\cos(n\theta_{kj})=-2\Lambda_{nj}q^{-\frac{wn}{2}}$$

for $n\neq 0$ and $\mu_{L_j}(1)=1$. Passing to the limit when $j\to\infty$ we get (4.2). $\square$

Corollary 4.2. *Let $\{\zeta_j(s)\}$ be an asymptotically very exact family of zeta functions with $\varepsilon_{w_{\mathrm{e}}}=1$ and let r_j be the order of zero of $\zeta_j(s)$ at $s=\frac{w_{\mathrm{e}}}{2}$. Then*

$$\lim_{j\to\infty}\frac{r_j}{\widetilde{d}_j}=0.$$

Proof. Suppose that $\limsup \frac{r_j}{\tilde{d}_j} = \varepsilon > 0$. Taking a subsequence we can assume that $\lim\limits_{j\to\infty} \frac{r_j}{\tilde{d}_j} = \varepsilon$. Taking a subsequence once again and using Proposition 3.12 we can assume that we are working with an asymptotically very exact sequence of L-functions $\{L_j(s)\} = \{L_{w_e j}(s)\}$ for which the same property concerning r_j holds.

By the previous theorem $\lim\limits_{j\to\infty} \mu_{L_j} = \mu_{\mathbf{lim}}$. Let us take an even continuous non-negative function $f(x) \in C[0, \pi]$ with the support contained in $[0, \frac{\varepsilon}{\alpha})$, where $\alpha = 2\max\left\{\int\limits_0^\pi \mu_{\mathbf{lim}}(x)\,dx, 1\right\}$ and such that $f(0) = 1$. We see that

$$\varepsilon \leqslant \lim_{j\to\infty} \mu_{L_j}(f(x)) = \int\limits_0^\pi f(x)\mu_{\mathbf{lim}}(x)\,dx \leqslant \frac{\varepsilon}{2},$$

so we get a contradiction. Thus the corollary is proven. □

Remark 4.3. It is easy to see that the same proof gives that the multiplicity of the zero at any particular point of the critical line grows asymptotically slower than d.

Remark 4.4. A thorough discussion of zero distribution results of similar type and their applications to various arithmetical problems can be found in [16].

4.2. Examples

Example 4.5 (Curves over finite fields)**.** In the case of curves over finite fields we recover the theorem 2.1 from [21]:

Corollary 4.6. *For an asymptotically exact family $\{X_i\}$ of curves over a finite field $\mathbb{F}_q$ the limit $\mu_{\{X_i\}} = \lim\limits_{i\to\infty} \mu_{X_i}$ is a continuous function given by an absolutely and uniformly convergent series*:

$$\mu_{\{X_i\}}(x) = 1 - \sum_{k=1}^{\infty} k\phi_k h_k(x),$$

where

$$h_k(x) = \frac{q^{k/2}\cos(kx) - 1}{q^k + 1 - 2q^{k/2}\cos(kx)}.$$

Proof. This follows from Theorem 4.1 together with the following series expansion:

$$\sum_{l=1}^{\infty} t^{-l}\cos(lkx) = \frac{t\cos(kx) - 1}{t^2 + 1 - 2t\cos(kx)}.$$ □

Example 4.7 (Varieties over finite fields)**.** We can not say much in this case since the zero distribution Theorem 4.1 applies only to L-functions. The only thing we get is that the multiplicity of zeroes on the line $\operatorname{Re} s = n - \frac{1}{2}$ divided by the sum of Betti numbers tends to zero (Corollary 4.2).

Example 4.8 (Elliptic curves over function fields)**.** Let us consider first asymptotically bad families of elliptic curves. We have the following corollary of Theorem 4.1.

Corollary 4.9. *For an asymptotically bad family of elliptic curves $\{E_i\}$ over function fields the zeroes of $L_{E_i}(s)$ become uniformly distributed on the critical line when $i \to \infty$.*

This result in the particular case of elliptic curves over the fixed field $\mathbb{F}_q(t)$ was obtained in [14]. In fact, unlike us, Michel gives an estimate for the difference $\mu_{E_i} - \mu_{\{E_i\}}$ in terms of the conductor n_{E_i}. It would be interesting to have such a bound in general.

Corollary 4.10. *For an asymptotically very exact family of elliptic curves $\{E_i/K_i\}$ obtained by a base change the limit $\mu_{\{E_i/K_i\}} = \lim_{i\to\infty} \mu_{E_i/K_i}$ is a continuous function given by an absolutely and uniformly convergent series*:

$$\mu_{\{E_i/K_i\}}(x) = 1 - \frac{2}{\nu + 4} \sum_{v,f} \phi_{v,f} f d_v \sum_{k=1}^{\infty} \frac{\alpha_v^k + \bar{\alpha}_v^k}{q^{fd_v k}} \cos(f d_v k x).$$

Corollary 4.11. *For a family of elliptic curves $\{E_i/K_i\}$ obtained by a base change*

$$\lim_{i\to\infty} \frac{r_i}{g_i} = 0.$$

Proof. By Proposition 3.18 any such family contains an asymptotically very exact subfamily so we can apply Corollary 4.2. □

Remark 4.12. For a fixed field K and elliptic curves over it a similar statement can be deduced from the bounds in [1]. However, in the case of the base change Brumer's bounds do not imply corollary 4.2. It would be interesting to see, what bounds one can get for the analytic ranks of individual elliptic curves when we vary the ground field K. Getting such a bound should be possible with a proper choice of a test function in the explicit formulae.

5. Brauer—Siegel type results

5.1. Limit zeta functions and the Brauer—Siegel theorem

Our approach to the Brauer—Siegel type results will be based on limit zeta functions.

Definition 5.1. Let $\{\zeta_k(s)\}$ be an asymptotically exact family of zeta functions. Then the corresponding limit zeta function is defined as

$$\zeta_{\mathbf{lim}}(s) = \exp\left(\sum_{f=1}^{\infty} \frac{\lambda_f}{f} q^{-fs}\right).$$

Remark 5.2. If $\zeta_k(s) = \zeta_{f_k}(s)$ are associated to some arithmetic or geometric objects f_k we will denote the limit zeta function simply by $\zeta_{\{f_k\}}(s)$.

Here are the first elementary properties of limit zeta functions:

Proposition 5.3. 1) *For an asymptotically exact family of zeta functions* $\{\zeta_k(s)\}$ *the series for* $\log \zeta_{\mathbf{lim}}(s)$ *is absolutely and uniformly convergent on compacts in the domain* $\operatorname{Re} s > \frac{w_{\mathbf{e}}}{2}$, *defining an analytic function there.*

2) *If a family is asymptotically very exact then* $\zeta_{\mathbf{lim}}(s)$ *is continuous for* $\operatorname{Re} s \geqslant \frac{w_{\mathbf{e}}}{2}$.

Proof. The first part of the proposition obviously follows from Proposition 3.5 together with Proposition 3.9.

By the definition of an asymptotically very exact family, the series for $\log \zeta_{\mathbf{lim}}(s)$ is uniformly and absolutely convergent for $\operatorname{Re} ys \geqslant \frac{w_{\mathbf{e}}}{2}$ so defines a continuous function in this domain. Thus the second part is proven. □

It is important to see to which extent limit zeta functions are the limits of the corresponding zeta functions over finite fields. The question is answered by the generalized Brauer—Siegel theorem. Before stating it let us give one more definition.

Definition 5.4. For an asymptotically exact family of zeta functions $\{\zeta_k(s)\}$ we call the limit $\lim\limits_{k\to\infty} \frac{\log \zeta_k(s)}{\widetilde{d}_k}$ the Brauer—Siegel ratio of this family.

Theorem 5.5 (The generalized Brauer—Siegel theorem)**.** *For any asymptotically exact family of zeta functions* $\{\zeta_k(s)\}$ *and any* s *with* $\operatorname{Re} s > \frac{w_{\mathbf{e}}}{2}$ *we have*

$$\lim_{k\to\infty} \frac{\log \zeta_{\mathbf{e},k}(s)}{\widetilde{d}_k} = \log \zeta_{\mathbf{lim}}(s).$$

If, moreover, $2 \operatorname{Re} s \notin \mathbb{Z}$, *then*

$$\lim_{k\to\infty} \frac{\log \zeta_k(s)}{\widetilde{d}_k} = \log \zeta_{\mathbf{lim}}(s).$$

The convergence is uniform in any domain $\frac{w_{\mathbf{e}}}{2} + \varepsilon < \operatorname{Re} s < \frac{w_{\mathbf{e}}+1}{2} - \varepsilon$, $\varepsilon \in \left(0, \frac{1}{2}\right)$.

Proof. To get the first statement we apply Proposition 3.9 and exchange the limit when $k \to \infty$ and the summation, which is legitimate since the series in question are absolutely and uniformly convergent in a small (but fixed) neighbourhood of s.

To get the second statement we apply Proposition 3.5, which gives us:

$$\lim_{k\to\infty} \frac{\log \zeta_{\mathbf{n},k}(s)}{\widetilde{d}_k} = 0.$$

Now the second part of the theorem follows from the first. □

Remark 5.6. It might be unclear, why we call such a statement the Brauer—Siegel theorem. We will see below in Subsection 5.3 that the above theorem indeed implies a natural analogue of the Brauer—Siegel theorem for curves and varieties over finite fields. It is quite remarkable that the proof of Theorem 5.5 is very easy (say, compared to the one in [21]) once one gives proper definitions.

Remark 5.7. Let us sketch another way to prove the generalized Brauer—Siegel theorem. It might seem unnecessarily complicated but it has the advantage of being applicable in the number field case when we no longer have the convergence of $\log L_k(s)$ for $\operatorname{Re} s > \frac{w}{2}$. We will deal with L-functions to simplify the notation. The main idea is to prove using Stark formula (Proposition 2.5 in the case of L-functions over finite fields) that $\frac{L_k'(s)}{L_k(s)} \leqslant C(\varepsilon) d_k$ for any s with $\operatorname{Re} s \geqslant \frac{w}{2} + \varepsilon$. Then we apply the Vitali theorem from complex analysis, which states that for a sequence of bounded holomorphic functions in a domain $\mathcal{D}$ it is enough to check the convergence at a set of points in $\mathcal{D}$ with a limit point in $\mathcal{D}$. This method is applied to Dedekind zeta functions in [26].

Remark 5.8. It is natural to ask, what is the behaviour of limit zeta or L-functions for $\operatorname{Re} s \leqslant \frac{w_{\mathbf{e}}}{2}$. Unfortunately nice properties of L-functions such as the functional equation or the Riemann hypothesis do not hold for $L_{\mathbf{lim}}(s)$. This can be seen already for families of zeta functions of curves. The point is that the behaviour of $L_{\mathbf{lim}}(s)$ might considerably differ from that of $\lim\limits_{k\to\infty} \frac{\log L_k(s)}{d_k}$ when we pass the critical line.

5.2. Behaviour at the central point

It seems reasonable to ask, what is the relation between limit zeta functions and the limits of zeta functions over finite fields on the critical line (that is for $\operatorname{Re} s = \frac{w_{\mathbf{e}}}{2}$). This relation seems to be rather complicated.

For example, one can prove that the limit $\lim\limits_{k\to\infty}\dfrac{1}{\widetilde{d}_k}\dfrac{\zeta_k'(1/2)}{\zeta_k(1/2)}$ is always 1 in families of curves (this can be seen from the functional equation), which is definitely not true for the value $\dfrac{\zeta_{\mathbf{lim}}'(1/2)}{\zeta_{\mathbf{lim}}(1/2)}$.

However, the knowledge of this relation is important for some arithmetic problems (see the example of elliptic surfaces in the next subsection). The general feeling is that for "most" families the statement of the generalized Brauer—Siegel theorem holds for $s=\frac{w_{\mathbf{e}}}{2}$. There are very few cases when we know it (see Section 7 for a discussion) and we, actually, can not even formulate this statement as a conjecture, since it is not clear what conditions on L-functions we should impose.

Still, in general one can prove the "easy" inequality. The term is borrowed from the classical Brauer—Siegel theorem from the number field case, where the upper bound is known unconditionally (and is easy to prove) and the lower bound is not proven in general (one has to assume either GRH or a certain normality condition on the number fields in question). This analogy does not go too far though for in the classical Brauer—Siegel theorem we work far from the critical line and here we study the behaviour of zeta functions on the critical line itself.

Let $\{\zeta_k(s)\}$ be an asymptotically exact family of zeta functions. We define r_k and c_k using the Taylor series expansion

$$\zeta_k(s)=c_k\Big(s-\frac{w_{\mathbf{e}}}{2}\Big)^{r_k}+O\Big(\Big(s-\frac{w_{\mathbf{e}}}{2}\Big)^{r_k+1}\Big).$$

Theorem 5.9. *For an asymptotically very exact family of zeta functions $\{\zeta_k(s)\}$ such that $\varepsilon_{w_{\mathbf{e}}}=1$ we have:*

$$\limsup_{k\to\infty}\frac{\log|c_k|}{\widetilde{d}_k}\leqslant\log\zeta_{\mathbf{lim}}\Big(\frac{w_{\mathbf{e}}}{2}\Big).$$

Proof. Replacing the family $\{\zeta_k(s)\}$ by the family $\{\zeta_{\mathbf{e},k}(s)\}$ we can assume that $w=w_{\mathbf{e}}$.

Let us write

$$\zeta_k(s)=c_k\Big(s-\frac{w}{2}\Big)^{r_k}F_k(s),$$

where $F_k(s)$ is an analytic function in the neighbourhood of $s=\frac{w}{2}$ such that $F_k\Big(\frac{w}{2}\Big)=1$. Let us put $s=\frac{w}{2}+\theta$, where $\theta>0$ is a small positive real number. We have

$$\frac{\log\zeta_k\Big(\frac{w}{2}+\theta\Big)}{\widetilde{d}_k}=\frac{\log c_k}{\widetilde{d}_k}+r_k\frac{\log\theta}{\widetilde{d}_k}+\frac{\log F_k\Big(\frac{w}{2}+\theta\Big)}{\widetilde{d}_k}.$$

To prove the theorem we will construct a sequence θ_k such that

(1) $\frac{1}{\widetilde{d}_k}\log\zeta_k\Big(\frac{w}{2}+\theta_k\Big)\to\log\zeta_{\mathbf{lim}}\Big(\frac{w}{2}\Big)$;

(2) $\frac{r_k}{\widetilde{d}_k}\log\theta_k\to 0$;

(3) $\liminf\frac{1}{\widetilde{d}_k}\log F_k\Big(\frac{w}{2}+\theta_k\Big)\geqslant 0$.

For each natural number N we choose $\theta(N)$ a decreasing sequence such that

$$\Big|\zeta_{\mathbf{lim}}\Big(\frac{w}{2}\Big)-\zeta_{\mathbf{lim}}\Big(\frac{w}{2}+\theta(N)\Big)\Big|<\frac{1}{2N}.$$

This is possible since $\zeta_{\mathbf{lim}}(s)$ is continuous for $\operatorname{Re} s\geqslant\frac{w}{2}$ by Proposition 5.3. Next, we choose a sequence $k'(N)$ with the property:

$$\Big|\frac{1}{d_k}\log\zeta_k\Big(\frac{w}{2}+\theta\Big)-\log\zeta_{\mathbf{lim}}\Big(\frac{w}{2}+\theta\Big)\Big|<\frac{1}{2N}$$

for any $\theta\in[\theta(N+1),\theta(N)]$ and any $k\geqslant k'(N)$. This is possible by Theorem 5.5. Then we choose $k''(N)$ such that

$$\frac{-r_k\log\theta(N+1)}{\widetilde{d}_k}\leqslant\frac{\theta(N)}{N}$$

for any $k\geqslant k''(N)$, which can be done thanks to corollary 4.2 that gives us for an asymptotically very exact family $\frac{r_k}{d_k}\to 0$. Finally, we choose an increasing sequence $k(N)$ such that $k(N)\geqslant\max(k'(N),k''(N))$ for any N.

Now, if we define $N=N(k)$ by the inequality $k(N)\leqslant k\leqslant k(N+1)$ and let $\theta_k=\theta(N(k))$, then from the conditions imposed on θ_k we automatically get (1) and (2). The delicate point is (3). We apply the Stark formula from Proposition 2.5 to get an estimate on $\Big(\log F_k\Big(\frac{w}{2}+\theta\Big)\Big)'$:

$$\frac{1}{\widetilde{d}_k}\Big(\log\zeta_k\Big(\frac{w}{2}+\theta\Big)-r_k\log\theta\Big)'=-\frac{\log q}{2\widetilde{d}_k}\sum_{i=0}^{w}\varepsilon_i d_i+$$
$$+\frac{1}{\widetilde{d}_k}\sum_{i=0}^{w-1}\varepsilon_i\sum_{L_i(\theta_{ij})=0}\frac{1}{\frac{w}{2}+\theta-\theta_{ij}}+\frac{1}{\widetilde{d}_k}\sum_{L_w(\theta_{wj})=0,\theta_{wj}\neq\frac{w}{2}}\frac{1}{\frac{w}{2}+\theta-\theta_{wj}}.$$

The first term on the right hand side is clearly bounded by $-\log q$ from below. The first sum involving L-functions is also bounded by a constant C_1 as can be seen applying the Stark formula to individual L-functions and then using Proposition 3.5. The last sum is non-negative. Thus, we see that $\frac{1}{\widetilde{d}_k}\Big(\log F_k\Big(\frac{w}{2}+\theta\Big)\Big)' \geqslant C$ for any small enough θ. From this and from the fact that $F_k\Big(\frac{w}{2}\Big)=1$ we deduce that

$$\frac{1}{\widetilde{d}_k}\log F_k\Big(\frac{w}{2}+\theta_k\Big) \geqslant C\theta_k \to 0.$$

This proves (3) as well as the theorem. ☐

Remark 5.10. In the case when $\varepsilon_{w_e}=-1$ we get an analogous statement with the opposite inequality.

Remark 5.11. The proof of the theorem shows the importance of "low" zeroes of zeta functions (that is zeroes close to $s=\frac{w}{2}$) in the study of the Brauer—Siegel ratio at $s=\frac{w}{2}$. The lack of control of these zeroes is the reason why we can not prove a lower bound on $\lim\limits_{k\to\infty}\frac{\log|c_k|}{\widetilde{d}_k}$.

Remark 5.12. If we restrict our attention to L-functions with integral coefficients (i. e. such that $\mathscr{L}(u)$ has integral coefficients), then we can see that the ratio $\frac{\log|c_k|}{\widetilde{d}_k}$ is bounded from below by $-w\log q$, at least for even w. This follows from a simple observation that if a polynomial with integral coefficients has a non-zero positive value at an integer point then this value is greater then or equal to one. One may ask whether there is a lower bound for arbitrary w and whether anything similar holds in the number field case.

5.3. Examples

Example 5.13 (Curves over finite fields)**.** First of all, let us show that the generalized Brauer—Siegel Theorem 5.5 implies the standard Brauer—Siegel theorem for curves over finite fields from [21].

Let h_X be the number of $\mathbb{F}_q$-rational pints on the Jacobian of X, i. e. $h_X=|\operatorname{Pic}^0_{\mathbb{F}_q}(X)|$.

Corollary 5.14. *For an asymptotically exact family of curves* $\{X_i\}$ *over a finite field* $\mathbb{F}_q$ *we have*:

$$\lim_{i\to\infty}\frac{\log h_{X_i}}{g_i}=\log q+\sum_{f=1}^{\infty}\phi_f\log\frac{q^f}{q^f-1}. \tag{5.1}$$

Proof. It is well known (cf. [23]) that for a curve X the number h_X can be expressed as $h_X=\mathscr{L}_X(1)$, where $\mathscr{L}_X(u)$ is the numerator of the zeta function of X (a polynomial in u). Using the functional equation for $\zeta_X(s)$ we see that $\log h_X = \log L_X(0) = \log L_X(1) + g\log q$.

The right hand side of (5.1) can be written as $\log q + 2\log\zeta_{\{X_i\}}(1)$, where $\zeta_{\{X_i\}}(s)$ is the limit zeta function $\zeta_{\mathbf{lim}}$ for the family of curves $\{X_i\}$ (the factor 2 appears from the definition of $\log\zeta_{\{X_i\}}(s)$, in which we divide by $2g$ and not by g). Thus, it is enough to prove that

$$\lim_{i\to\infty}\frac{\log L_{X_i}(1)}{2g_i} = \log\zeta_{\{X_i\}}(1).$$

This follows immediately from the first equality of Theorem 5.5. □

Using nearly the same proof we can obtain one more statement about the asymptotic behaviour of invariants of function fields. To formulate it we will need to define the so called Euler—Kronecker constants of a curve X (see [6]):

Definition 5.15. Let X be a curve over a finite field $\mathbb{F}_q$ and let

$$\frac{\zeta_X'(s)}{\zeta_X(s)} = -(s-1)^{-1} + \gamma_X^0 + \gamma_X^1(s-1) + \gamma_X^2(s-1)^2 + \ldots$$

be the Taylor series expansion of $\dfrac{\zeta_X'(s)}{\zeta_X(s)}$ at $s=1$. Then $\gamma_X = \gamma_X^0$ is called the Euler—Kronecker constant of X and γ_X^k, $k\geqslant 1$ are called the higher Euler—Kronecker constants.

We also define the asymptotic Euler—Kronecker constants $\gamma_{\{X_i\}}^k$ from:

$$\frac{\zeta_{\{X_i\}}'(s)}{\zeta_{\{X_i\}}(s)} = \gamma_{\{X_i\}}^0 + \gamma_{\{X_i\}}^1(s-1) + \gamma_{\{X_i\}}^2(s-1)^2 + \ldots$$

($\zeta_{\{X_i\}}(s)$ is holomorphic and non-zero at $s=1$ so its logarithmic derivative has no pole at this point).

The following result generalizes theorem 2 from [6]:

Corollary 5.16. *For an asymptotically exact family of curves $\{X_i\}$ we have*

$$\lim_{i\to\infty}\frac{\gamma_{X_i}^k}{g_i} = \gamma_{\{X_i\}}^k$$

for any non-negative integer k. In particular,

$$\lim_{i\to\infty}\frac{\gamma_{X_i}}{g_i} = -\sum_{f=1}^{\infty}\frac{\phi_f f\log q}{q^f-1}.$$

Proof. We apply the first equality from Theorem 5.5. Using the explicit expression for the negligible part of zetas as $(1-q^{-s})(1-q^{1-s})$, we see that

$$\lim_{i\to\infty}\frac{\log\zeta_{X_i}(s)}{2g_i}=\log\zeta_{\{X_i\}}(s)$$

for any s, such that $\operatorname{Re}s>\frac{1}{2}$ and $s\neq 1+\frac{2\pi k}{\log q}$, $k\in\mathbb{Z}$ and the convergence is uniform in $a<|s-1|<b$ for small enough a and b. We use the Cauchy integral formula to get the statement of the corollary. □

Remark 5.17. It seems not completely uninteresting to study the behaviour of γ_X^k "on the finite level", i. e. to try to obtain bounds on γ_X^k for an individual curve X. This was done in [6] for γ_X. In the general case the explicit version of the generalized Brauer—Siegel theorem from [13] might be useful.

Remark 5.18. It is worth noting that the above corollaries describe the relation between $\log\zeta_{X_i}(s)$ and $\log\zeta_{\{X_i\}}(s)$ near the point $s=1$. The original statement of Theorem 5.5 is stronger since it gives this relation for all s with $\operatorname{Re}s>\frac{1}{2}$.

Example 5.19 (Varieties over finite fields)**.** Just as for curves, for varieties over finite fields we can get similar corollaries concerning the asymptotic behaviour of $\zeta_{X_i}(s)$ close to $s=d$. We give just the statements, since the proofs are nearly the same as before.

The following result is the Brauer—Siegel theorem for varieties proven in [24].

Corollary 5.20. *For an asymptotically exact family of varieties $\{X_i\}$ of dimension n over a finite field $\mathbb{F}_q$ we have*:

$$\lim_{i\to\infty}\frac{\log|\varkappa_i|}{b(X_i)}=\sum_{f=1}^{\infty}\phi_f\log\frac{q^{fn}}{q^{fn}-1},$$

where $\varkappa_i=\operatorname{Res}_{s=d}\zeta_{X_i}(s)$.

In the next corollary we use the same definition of the Euler—Kronecker constants for varieties over finite fields as in the previous example for curves:

Corollary 5.21. *For an asymptotically exact family of varieties $\{X_i\}$ of dimension n we have* $\lim\limits_{i\to\infty}\frac{\gamma_{X_i}^k}{b(X_i)}=\gamma_{\{X_i\}}^k$ *for any k. In particular,*

$$\lim_{i\to\infty}\frac{\gamma_{X_i}}{b(X_i)}=-\sum_{f=1}^{\infty}\frac{\phi_f f\log q}{q^{fn}-1}.$$

Example 5.22 (Elliptic curves over function fields)**.** Let us recall first the Brauer—Siegel type conjectures for elliptic curves over function fields due to Hindry—Pacheko [5] and Kunyavskii—Tsfasman [11].

For an elliptic curve E/K, $K=\mathbb{F}_q(X)$ we define $c_{E/K}$ and $r_{E/K}$ from $L_{E/K}(s)=c_{E/K}(s-1)^{r_{E/K}}+o((s-1)^{r_{E/K}})$. The invariants $r_{E/K}$ and $c_{E/K}$ are important from the arithmetical point of view, since the geometric analogue of the Birch and Swinnerton-Dyer conjecture predicts that $r_{E/K}$ is equal to the rank of the group of K-rational points of E/K and $c_{E/K}$ can be expressed via the order of the Shafarevich—Tate group, the covolume of the Mordell—Weil lattice (the regulator) and some other quantities related to E/K which are easier to control.

Conjecture 5.23 (Hindry—Pacheko). *Let E_i run through a family of pairwise non-isomorphic elliptic curves over a fixed function field K. Then*

$$\lim_{i\to\infty}\frac{\log|c_{E_i/K}|}{h(E_i)}=0,$$

where $h(E_i)$ is the logarithmic height of E_i.

Remark 5.24. We could have divided $\log|c_{E_i/K}|$ by n_{E_i} in the statement of the above conjecture since $h(E_i)$ and n_{E_i} have essentially the same order of growth.

Conjecture 5.25 (Kunyavskii—Tsfasman). *For a family of elliptic curves $\{E_i/K_i\}$ obtained by a base change we have:*

$$\lim_{i\to\infty}\frac{\log|c_{E_i/K_i}|}{g_{K_i}}=-\sum_{v\in X,f\geqslant 1}\phi_{v,f}\log\frac{|E_v(\mathbb{F}_{\mathrm{N}v^f})|}{\mathrm{N}v^f}.$$

One can see that the above conjectures are both the statements of the type considered in the Subsection 5.2. It is quite obvious for the first conjecture and for the second conjecture we have to use the explicit expression for the limit L-function:

$$\log L_{\{E_i/K_i\}}(s)=-\frac{1}{\nu+4}\sum_{v,f}\phi_{v,f}\log\bigl(1-(\alpha_v^f+\bar{\alpha}_v^f)\mathrm{N}v^{-fs}+\mathrm{N}v^{f(1-2s)}\bigr).$$

One can unify these two conjectures as follows:

Conjecture 5.26. *For an asymptotically very exact family of elliptic curves over function fields $\{E_i/K_i\}$ we have:*

$$\lim_{i\to\infty}\frac{\log|c_{E_i/K_i}|}{d_i}=\log L_{\{E_i/K_i\}}(1),$$

where $d_i=n_{E_i}+4g_{K_i}-4$ is the degree of $L_{E_i/K_i}(s)$.

We are, however, sceptical about this conjecture holding for all families of elliptic curves. Theorems 5.5 and 5.9 imply the following result (a particular case of which was stated in [25]) in the direction of the above conjectures:

Theorem 5.27. *For an asymptotically very exact family of elliptic curves* $\{E_i/K_i\}$ *the following identity holds*:

$$\lim_{i\to\infty}\frac{\log L_{E_i/K_i}(s)}{d_i}=\log L_{\{E_i/K_i\}}(s),$$

for $\operatorname{Re} s>1$. *Moreover*,

$$\lim_{i\to\infty}\frac{\log|c_{E_i/K_i}|}{g_i}\leqslant\log L_{\{E_i/K_i\}}(1).$$

Remark 5.28. If we consider split families of elliptic curves (i. e. $E_i=E\times X_i$, where $E/\mathbb{F}_q$ is a fixed elliptic curve) then the proof of theorem 2.1 from [11] gives us that the question about the behaviour of $L_{E_i/X_i}(s)$ at $s=1$ translates into the same question concerning the behaviour of $\zeta_{X_i}(s)$ on the critical line. For example, if the curve E is supersingular, then Conjecture 5.26 holds if and only if

$$\lim_{i\to\infty}\frac{\log|\zeta_{X_i}(1/2)|}{g_i}=\log\zeta_{\{X_i\}}\Bigl(\frac{1}{2}\Bigr)$$

(where $\zeta_{X_i}\Bigl(\frac{1}{2}\Bigr)$ is understood as the first non-zero coefficient of the Taylor series expansion of $\zeta_{X_i}(s)$ at $s=\frac{1}{2}$). So, to prove the simplest case of Conjecture 5.26 we have to understand the asymptotic behaviour of zeta functions of curves over finite fields on the critical line.

6. Basic inequalities

The goal of this section is to prove various versions of the basic inequality which can be seen as a generalization of the Drinfeld—Vlăduţ inequality for the number of points on curves over finite fields. We will start with the case of L-functions, where a little more can be said. Next, we will prove a weaker result in the case of zeta functions.

6.1. Basic inequality for L-functions

Our goal here is to prove the following theorem, generalizing the basic inequality from [20].

Theorem 6.1. *Assume we have an asymptotically exact family* $\{L_k(s)\}$ *of L-functions of weight* w *or an asymptotically exact family of zeta functions* $\{\zeta_i(s)\}$ *with* $\zeta_{\mathbf{e},i}(s)$ *being an L-function of weight* w *for any* i. *Then for any* $b\in\mathbb{N}$ *the following inequality holds*:

$$\sum_{j=1}^{b}\Bigl(1-\frac{j}{b+1}\Bigr)\lambda_j q^{-\frac{wj}{2}}\leqslant\frac{1}{2}. \tag{6.1}$$

Proof. Using Proposition 3.9 one immediately sees that it is enough to prove the statement of the theorem for L-functions.

As in the proof for curves our main tool will be the so called Drinfeld inequality. We take an L-function $L(s)$ and let $\alpha_i = q^{\frac{w}{2}}\rho_i$, where ρ_i are the roots of $\mathscr{L}(u)$, so that $|\alpha_i| = 1$. For any α_i we have

$$0 \leqslant |\alpha_i^b + \alpha_i^{b-1} + \ldots + 1|^2 = (b+1) + \sum_{j=1}^{b}(b+1-j)(\alpha_i^j + \alpha_i^{-j}).$$

Thus

$$b+1 \geqslant -\sum_{j=1}^{b}(b+1-j)(\alpha_i^j + \alpha_i^{-j}).$$

We sum the inequalities for $i = 1, \ldots, d$. Since the coefficients of $\mathscr{L}(u)$ are real we note that $\sum_{i=1}^{d}\alpha_i^j = \sum_{i=1}^{d}\alpha_i^{-j}$. From (3.1) we see that $\Lambda_j = -q^{wj}\sum_{i=1}^{d}\rho_i^j$. Putting it together we get:

$$d(b+1) \geqslant 2\sum_{j=1}^{b}(b+1-j)\Lambda_j q^{-\frac{wj}{2}}.$$

Now, we let vary $L_k(s)$ so that $d_k \to \infty$ and obtain the stated inequality. ☐

Unfortunately, we are unable to say anything more in general without the knowledge of some additional properties of λ_j. However, the next corollary shows that sometimes we can do better.

Corollary 6.2. *If a family $\{L_k(s)\}$ is asymptotically exact then*

$$\sum_{j=1}^{\infty}\lambda_j q^{-\frac{wj}{2}} \leqslant \frac{1}{2},$$

provided one of the following conditions holds:

1) *either it is asymptotically very exact or*

2) $\lambda_j \geqslant 0$ *for any* j.

Proof. To prove the statement of the corollary under the first assumption we choose an $\varepsilon > 0$ and $b' \in \mathbb{N}$ such that the sum

$$\sum_{j=b'+1}^{\infty}|\lambda_j| q^{-\frac{wj}{2}} < \varepsilon.$$

Taking $b'' > b$, we apply the inequality from Theorem 6.1 with $b = b''$. We get:

$$\begin{aligned}
\frac{1}{2} &\geqslant \sum_{j=1}^{b''}\Big(1-\frac{j}{b''+1}\Big)\lambda_j q^{-\frac{wj}{2}} = \\
&= \sum_{j=1}^{b'}\Big(1-\frac{j}{b''+1}\Big)\lambda_j q^{-\frac{wj}{2}} + \sum_{j=b'+1}^{b''}\Big(1-\frac{j}{b''+1}\Big)\lambda_j q^{-\frac{wj}{2}} \geqslant \\
&\geqslant \sum_{j=1}^{b'}\Big(1-\frac{j}{b''+1}\Big)\lambda_j q^{-\frac{wj}{2}} - \sum_{j=b'+1}^{b''}\Big|1-\frac{j}{b''+1}\Big||\lambda_j| q^{-\frac{wj}{2}} \geqslant \\
&\geqslant \sum_{j=1}^{b'}\Big(1-\frac{j}{b''+1}\Big)\lambda_j q^{-\frac{wj}{2}} - \varepsilon.
\end{aligned}$$

We now let tend b'' to $+\infty$ and get $\sum_{j=1}^{b'} \lambda_j q^{-\frac{wj}{2}} - \varepsilon \leqslant \frac{1}{2}$. Then, passing to the limit when $b' \to \infty$, we see that the first part of the corollary is true.

To prove the statement under the second condition we use the same trick. We take $b' \in \mathbb{N}$ such that $\frac{b}{b'+1} < \varepsilon$. Then we apply Theorem 6.1 with $b = b'$ and notice that the sum only decreases when we drop the part $\sum_{j=b+1}^{b'}\Big(1-\frac{j}{b'+1}\Big)\lambda_j q^{-\frac{wj}{2}}$ since $\lambda_j \geqslant 0$:

$$\frac{1}{2} \geqslant \sum_{j=1}^{b}\Big(1-\frac{j}{b'+1}\Big)\lambda_j q^{-\frac{wj}{2}} \geqslant \sum_{j=1}^{b}\Big(1-\frac{j}{b'+1}\Big)\lambda_j q^{-\frac{wj}{2}} \geqslant \sum_{j=1}^{b'}(1-\varepsilon)\lambda_j q^{-\frac{wj}{2}}$$

This gives the second part of the corollary. □

Corollary 6.3. *Any asymptotically exact family of L-functions, satisfying $\lambda_j \geqslant 0$ for any j, is asymptotically very exact.*

Remark 6.4. The statements of both of the corollaries are obviously still true if one assumes that $\lambda_j \geqslant 0$ for all but a finite number of $j \in \mathbb{N}$.

Remark 6.5. Using Theorem 4.1 one can give another proof of the basic inequality for asymptotically very exact families of L-functions. Indeed, all the measures defined by (4.1) are non-negative. Thus the limit measure $\mu_{\mathbf{lim}}$ must have a non-negative density at any point, in particular at $x = 0$. This gives us exactly the basic inequality. In this way we get an interpretation of the difference between the right hand side and the left hand side of the basic inequality as “the asymptotic number of zeroes of $L_j(s)$, accumulating at $s = \frac{w}{2}$”.

In fact, using the same reasoning as before, we get a family of inequalities (which are interesting when not all the coefficients λ_f are non-negative):

$$\sum_{k=1}^{\infty} \lambda_k \cos(kx) q^{-\frac{wk}{2}} \leqslant \frac{1}{2}$$

for any $x \in \mathbb{R}$.

6.2. Basic inequality for zeta functions

We have noticed before that even in the case of L-functions we do not get complete results unless we assume that our family is asymptotically very exact or all the coefficients λ_f are positive. While working with zeta functions we face the same problem. However, we will deal with it in a different way for no general lower bound on the sums of the type (6.1) seems to be available and such a lower bound would definitely be necessary since zeta functions are products of L-functions both in positive and in negative powers.

Theorem 6.6. *Let $\{\zeta_k(s)\}$ be an asymptotically exact family of zeta functions. Then for any real s with $\frac{w_{\mathbf{e}}}{2} < s < \frac{w_{\mathbf{e}}+1}{2}$ we have:*

$$-\sum_{i=0}^{w_{\mathbf{e}}} \frac{\delta_i}{q^{s-i/2} - \varepsilon_i} \leqslant -\frac{1}{\log q} \frac{\zeta'_{\mathbf{lim}}(s)}{\zeta_{\mathbf{lim}}(s)} \leqslant \sum_{i=0}^{w_{\mathbf{e}}} \frac{\delta_i}{q^{s-i/2} + \varepsilon_i},$$

or, more explicitly,

$$-\sum_{i=0}^{w_{\mathbf{e}}} \frac{\delta_i}{q^{s-i/2} - \varepsilon_i} \leqslant \sum_{j=1}^{\infty} \lambda_j q^{-sj} \leqslant \sum_{i=0}^{w_{\mathbf{e}}} \frac{\delta_i}{q^{s-i/2} + \varepsilon_i}.$$

Proof. First of all, Proposition 3.9 implies that it is enough to prove the inequality in the case when $\zeta_k(s) = \zeta_{\mathbf{e},k}(s)$ and $w = w_{\mathbf{e}}$.

Let us write the Stark formula from Proposition 2.5:

$$\frac{1}{\log q} \frac{\zeta'(s)}{\zeta(s)} = \sum_{i=0}^{w} \varepsilon_i \sum_{j=1}^{d_i} \frac{1}{q^s \rho_{ij} - 1}.$$

We notice that all the terms are real for real s and

$$R(r, \theta) = \operatorname{Re} \frac{re^{i\theta}}{1 - re^{i\theta}} = \frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2}.$$

Applying this relation we see that

$$\frac{1}{\log q} \frac{\zeta'(s)}{\zeta(s)} = \sum_{i=0}^{w} \varepsilon_i \sum_{j=1}^{d_i} R(q^{i/2-s}, \theta_{ij}),$$

where $\rho_{kj} = q^{-\frac{k}{2}} e^{i\theta_{kj}}$.

For $0<r<1$ we have the bounds on $R(r,\theta)$:

$$-\frac{r}{1+r} \leqslant R(r,\theta) \leqslant \frac{r}{1-r}.$$

For $\varepsilon \in \{\pm 1\}$ this implies

$$-\frac{1}{1/r+\varepsilon} \leqslant \varepsilon R(r,\theta) \leqslant \frac{1}{1/r+\varepsilon}.$$

From this we deduce that for s with $\frac{w}{2} < s < \frac{w+1}{2}$ the following inequality holds

$$-\sum_{i=0}^{w} \frac{d_i}{q^{s-i/2}-\varepsilon_i} \leqslant \frac{-1}{\log q}\frac{\zeta'(s)}{\zeta(s)} \leqslant \sum_{i=0}^{w} \frac{d_i}{q^{s-i/2}+\varepsilon_i}. \tag{6.2}$$

The next step is to use Theorem 5.5. For any s in the interval $\left(\frac{w}{2}, \frac{w+1}{2}\right)$ it gives that

$$\lim_{k\to\infty} \frac{-1}{\widetilde{d}_k \log q} \cdot \frac{\zeta_k'(s)}{\zeta_k(s)} = \sum_{j=1}^{\infty} \lambda_j q^{-\frac{sj}{2}}.$$

Dividing (6.2) by $\widetilde{d}_k$, passing to the limit and using the previous equality we get the statement of the theorem. □

Corollary 6.7. 1) *If $\varepsilon_{w_{\mathrm{e}}}=1$ and either the family is asymptotically very exact or $\lambda_j \geqslant 0$ for all j then*

$$\sum_{j=1}^{\infty} \lambda_j q^{-\frac{w_{\mathrm{e}} j}{2}} \leqslant \sum_{i=0}^{w_{\mathrm{e}}} \frac{\delta_i}{q^{(w_{\mathrm{e}}-i)/2}+\varepsilon_i}$$

2) *If $\varepsilon_{w_{\mathrm{e}}}=-1$ and either the family is asymptotically very exact or $\lambda_j \leqslant 0$ for all j then*

$$-\sum_{i=0}^{w_{\mathrm{e}}} \frac{\delta_i}{q^{(w_{\mathrm{e}}-i)/2}-\varepsilon_i} \leqslant \sum_{j=1}^{\infty} \lambda_j q^{-\frac{w_{\mathrm{e}} j}{2}}.$$

Proof. Let us suppose that $\varepsilon_{w_{\mathrm{e}}}=1$ (the other case is treated similarly). For an asymptotically very exact family for any $\varepsilon>0$ we can choose $N>0$ such that $\sum_{j>N}^{\infty} |\lambda_j| q^{-\frac{w_{\mathrm{e}} j}{2}} < \varepsilon$. Thus both for a very exact family and for a family with $\lambda_j \geqslant 0$ for all j we have

$$\sum_{j=1}^{N} \lambda_j q^{-sj} \leqslant \sum_{i=0}^{w_{\mathrm{e}}} \frac{\delta_i}{q^{s-i/2}+\varepsilon_i} + \varepsilon$$

for any real s with $\frac{w_{\mathrm{e}}}{2} < s < \frac{w_{\mathrm{e}}+1}{2}$. Passing to the limit when $s \to \frac{w_{\mathrm{e}}}{2}$ we get the statement of the corollary. □

Corollary 6.8. *Any asymptotically exact family, such that*

$$\varepsilon_{w_e} \operatorname{sign}(\lambda_j) = 1$$

for any j, is asymptotically very exact.

Remark 6.9. Though the Corollary 6.7 implies the Corollary 6.2, the basic inequality for L-functions given by Theorem 6.1 is different from the one obtained by application of Theorem 6.6.

6.3. Examples

Example 6.10 (Curves over finite fields)**.** For curves over finite fields we obtain once again the classical basic inequality from [20]:

$$\sum_{j=1}^{\infty} 2\lambda_j q^{-\frac{j}{2}} = \sum_{m=1}^{\infty} \frac{m\phi_m}{q^{m/2}-1} \leqslant 1.$$

Of course, this is not an interesting example for us, since we used this inequality as our initial motivation.

Example 6.11 (Varieties over finite fields)**.** In a similar way, for varieties over finite fields we get the inequality from [12, (8.8)]:

$$\sum_{m=1}^{\infty} \frac{m\phi_m}{q^{(2d-1)m/2}-1} \leqslant (q^{(2d-1)/2}-1)\left(\frac{\beta_1}{2} + \sum_{2|i} \frac{\beta_i}{q^{(i-1)/2}+1} + \sum_{2\nmid i} \frac{\beta_i}{q^{(i-1)/2}-1}\right).$$

With more efforts one can reprove most (if not all) of the inequalities from [12, (8.8)] in our general context of zeta functions, since the main tools used in [12] are the explicit formulae. However, we do not do it here as for the moment we are unable see any applications it might have to particular examples of zeta functions.

Example 6.12 (Elliptic curves over function fields)**.** The case of asymptotically bad families is trivial: we do not obtain any interesting results here since all $\lambda_j = 0$.

Let us consider the base change case. Let us take an asymptotically very exact family of elliptic curves obtained by a base change (by Proposition 3.18 any family obtained by a base change is asymptotically very exact, provided $\operatorname{char} \mathbb{F}_q \neq 2, 3$). We can apply Corollary 6.2 to obtain that $\sum\limits_{j=1}^{\infty} \lambda_j q^{-j/2} \leqslant \frac{1}{2}$. Using (3.3), one can rewrite it using $\phi_{v,m}$ as follows:

$$\sum_{v,m} \frac{m d_v \phi_{v,m} (\alpha_v^m + \bar{\alpha}_v^m) q^{-md_v}}{1-(\alpha_v^m + \bar{\alpha}_v^m) q^{-md_v}} \leqslant \frac{\nu+4}{2}$$

(here $\nu = \lim\limits_{i\to\infty} \dfrac{n_{E_i/K_i}}{g_{K_i}}$).

7. Open questions and further research directions

In this section we would like to gather together the questions which naturally arise in the connection with the previous sections. Let us start with some general questions. First of all:

Question 7.1. To which extent the formal zeta and L-functions defined in Section 2 come from geometry?

One can make it precise in several ways. For example, it is possible to ask whether any L-function of weight w, such that $\mathscr{L}(u)$ has integral coefficients is indeed the characteristic polynomial of the Frobenius automorphism acting on the w-th cohomology group of some variety $V/\mathbb{F}_q$. A partial answer to this question when $w=1$ is provided by the Honda—Tate theorem on abelian varieties [19].

Question 7.2. Describe the set $\{(\lambda_1, \lambda_2, \dots)\}$ for asymptotically exact (very exact) families of zeta functions (L-functions).

There are definitely some restrictions on this set, namely those given by various basic inequalities (Theorems 6.1 and 6.6, Remark 6.5). It would be interesting to see whether there are any others. We emphasize that the problem is not of arithmetic nature since we do not assume that the coefficients of polynomials, corresponding to L-functions, are integral. It would be interesting to see what additional restrictions the integrality condition on the coefficients of $\mathscr{L}(u)$ might give. Note that, using geometric constructions, Tsfasman and Vlăduţ [21] proved that the sets of parameters λ_f, satisfying $\lambda_f \geqslant 0$ for any f and the basic inequality are all realized when q is a square and $w=1$. This implies the same statement for L-functions with arbitrary q and w. However, our new L-function might no longer have integral coefficients.

Question 7.3. How many asymptotically good (very good) families are there among all asymptotically exact (very exact) families?

The “how many” part of the question should definitely be made more precise. One way to do this is to consider the set V_g of the vectors of coefficients of polynomials corresponding to L-functions of degree d and its subset $V_d(f,a,b)$ consisting of the vectors of coefficients of polynomials corresponding to L-functions with $a < \frac{\Lambda_f}{d} < b$. A natural question is whether the ratio of the volume of $V_d(f,a,b)$ to the volume of V_g has a limit when $d \to \infty$ and what this limit is. See [2] for some information about V_d. The question is partly justified by the fact that it is difficult to construct asymptotically good families of curves. We would definitely like to know why.

Let us now ask some questions concerning the concrete results on zeta and L-functions proven in the previous sections.

Question 7.4. Is it true that the generalized Brauer—Siegel Theorem 5.5 holds on the critical line for some (most) asymptotically very exact families?

It is sure that without the additional arithmetic conditions on the family the statement does not hold. The most interesting families here are the families of elliptic curves over function fields considered in Subsection 5.3 due to the arithmetic applications. An example of a family of elliptic surfaces for which the statement holds is given in [5]. It is interesting to look at some other particular examples of families of curves over finite fields where the corresponding zeta functions are more or less explicitly known. These include the Fermat curves [10] and the Jacobi curves [9].

Some examples we know to support the positive answer to the above question come from the number field case. It is known that there exists a sequence $\{d_i\}$ in $\mathbb{N}$ of density at least $\frac{1}{3}$ such that

$$\lim_{i\to\infty} \frac{\log \zeta_{\mathbb{Q}(\sqrt{d_i})}\left(\frac{1}{2}\right)}{\log d_i} = 0$$

(cf. [8]). The techniques of the evaluation of mollified moments of Dirichlet L-functions used in that paper is rather involved. It would be interesting to know whether one can obtain analogous results in the function field case. The related questions in the function field case are studied in [10]. It is not clear whether the results on the one level densities for zeroes obtained there can be applied to the question of finding a lower bound on $\frac{\log |c_i|}{d_i}$ for some positive proportion of fields (both in the number field and in the function field cases).

Question 7.5. Prove the generalized Brauer—Siegel Theorem 5.5 with an explicit error term.

This was done for curves over finite fields in [13] and looks quite feasible in general. It is also worth looking at particular applications that such a result might have, in particular one may ask what bounds on the Euler—Kronecker constants it gives.

Question 7.6. How to characterize measures corresponding to asymptotically very exact families?

This was done in [21] for families such that $\lambda_f \geqslant 0$ for all f. The general case remains open.

Question 7.7. Estimate the error term in Theorem 4.1.

As it was mentioned before, in the case of elliptic curves over $\mathbb{F}_q(t)$ the estimates were carried out in [14].

Question 7.8. Find explicit bounds on the orders of zeroes of L-functions on the line $\mathrm{Re}\, s = \frac{w}{2}$.

The Corollary 4.2 gives that the ratio $\frac{r_i}{d_i} \to 0$ for asymptotically very exact families (here r_i is the multiplicity of the zero). In a particular case of elliptic curves over a fixed function field Brumer in [1] gives a bound which grows asymptotically slower than the conductor. Using explicit formulae with a proper choice of test functions, it should be possible to give such upper bounds for families obtained by a base change if not in general.

Let us finally ask a few more general questions.

Question 7.9. How can one apply the results of this paper to get the information about the arithmetic or geometric properties of the objects to which L-functions are associated?

We carried out this task (to a certain extent) in the case of curves and varieties over finite fields and elliptic curves over function fields. Additional examples are more than welcome.

The last but not least:

Question 7.10. What are the number field analogues of the results obtained in this paper?

It seems that most of the results can be generalized to the framework of the Selberg class (as described, for example, in [7, Chapter 5]), subject to imposing some additional hypothesis (such as the Generalized Riemann Hypothesis, the Generalized Ramanujan Conjectures, etc.). Of course, one will have to overcome quite a lot of analytical difficulties on the way (compare, for example, [21] and [22]).

We hope to return to this interesting and promising subject later on.

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Uniform distribution of zeroes of L-functions of modular forms

Abstract. We prove under GRH that zeros of L-functions of modular forms of level N and weight k become uniformly distributed on the critical line when $N+k\to\infty$.

1. Introduction

It is well known that zeroes of L-functions contain an important information about the arithmetic properties of the objects to which these L-functions are associated. The question about the distribution of these zeroes on the critical line was studied by many authors. This problem can be looked upon from many angles (the proportion of zeroes on the critical line, low zeroes, zero spacing, etc.).

In this paper we study the distribution of zeroes of L-functions on the critical line when we let vary the modular form to which the L-function is associated. The same question was considered by S. Lang in [4] and M. Tsfasman and S. Vlăduţ in [8] for the Dedekind zeta function of number fields.

Let $f(z)$ be a holomorphic cusp of weight $k=k_f$ for the group $\Gamma_0(N)$ such that $f(z)=\sum\limits_{n=1}^{\infty} a_n n^{(k-1)/2}e^{2\pi i n z}$ is its normalized Fourier expansion at the cusp ∞. We suppose that $f(z)$ is a primitive form in the sense of Atkin—Lehner [1] (it is a new form and a normalized eigen form for all Hecke operators), so $L_f(s)$ can be defined by the Euler product

$$L_f(s)=\prod_{p\mid N}(1-a_p p^{-s})^{-1}\prod_{p\nmid N}(1-a_p p^{-s}+p^{-2s})^{-1}.$$

We denote by α_p and $\bar\alpha_p$ the two conjugate roots of the polynomial $1-a_p p^{-s}+p^{-2s}$. Deligne has shown (see [2]) that $|\alpha_p|=|\bar\alpha_p|=1$ for $p\nmid N$ (the Ramanujan—Peterson conjecture). On the other hand, one knows (see [1]) that for $p\mid N$ we have $|a_p|\leqslant 1$.

If we define the gamma factor by

$$\gamma_f(s)=\pi^{-s}\Gamma\Big(\frac{s+(k-1)/2}{2}\Big)\Gamma\Big(\frac{s+(k+1)/2}{2}\Big)=c_k(2\pi)^{-s}\Gamma\Big(s+\frac{k-1}{2}\Big)$$

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with $c_k = 2^{(3-k)/2}\sqrt{\pi}$, then the function $\Lambda(s) = N^{s/2}\gamma_f(s)L_f(s)$ is entire and satisfies the functional equation $\Lambda(s) = w\Lambda(1-s)$ with $w = \pm 1$. The Generalized Riemann Hypothesis (GRH) for L-function of modular forms states that all the non-trivial zeroes of these L-functions lie on the critical line $\operatorname{Re} s = \frac{1}{2}$. Throughout the paper we assume that GRH is true.

The analytic conductor q_f (see [3]) is defined as

$$q_f = N\Big(\frac{k-1}{2} + 3\Big)\Big(\frac{k+1}{2} + 3\Big) \sim \frac{Nk^2}{4},$$

when $k \to \infty$. We will use the last expression (or, more precisely, its logarithm minus a constant) as a weight in all the zero sums in the paper.

To each $f(z)$ we can associate the measure

$$\Delta_f := \frac{2\pi}{\log q_f} \sum_{L_f(\rho)=0} \delta_{t(\rho)},$$

where $t(\rho) = \frac{1}{i}\Big(\rho - \frac{1}{2}\Big)$ and ρ runs through all non-trivial zeroes of $L_f(s)$; here δ_a denotes the atomic (Dirac) measure at a. Since we suppose that GRH is true, Δ_f is a discrete measure on $\mathbb{R}$. Moreover, it can easily be seen that Δ_f is a measure of slow growth (see below).

Our main result is the following one:

Theorem 1.1. *Assuming GRH, for any family $\{f_j(z)\}$ of primitive forms with $q_{f_j} \to \infty$ the limit*

$$\Delta = \lim_{j\to\infty} \Delta_j = \lim_{j\to\infty} \Delta_{f_j}$$

exists in the space of measures of slow growth on $\mathbb{R}$ and is equal to the measure with density 1 (*i. e. dx*).

2. Proof of Theorem 1.1

Our method of the proof will, roughly speaking, follow that of [8], where a similar question is treated in the case of Dedekind zeta functions. It will even be simplier in our case due to the fact that the family we consider is "asymptotically bad".

Let us recall a few facts and definitions from the theory of distribution. We will use [7] as our main reference. Recall that the Schwartz space $\mathscr{S} = \mathscr{S}(\mathbb{R})$ is the space of all real valued infinitely differentiable rapidly decreasing functions on $\mathbb{R}$ (i. e. $\phi(x)$ and any its derivative go to 0 when $|x| \to \infty$ faster then any power of $|x|$). The space $\mathscr{D}(\mathbb{R})$ is defined to be the space of all real valued infinitely differentiable functions with compact support on $\mathbb{R}$. Both $\mathscr{S}(\mathbb{R})$ and $\mathscr{D}(\mathbb{R})$ are equipped with the structures of topological vector spaces.

The space $\mathscr{D}'$ (resp. $\mathscr{S}'$), topologically dual to $\mathscr{D}$ (resp. $\mathscr{S}$) is called the space of distribution (resp. tempered distributions). We also define the space of measures $\mathscr{M}$ as the topological dual of the space of real valued continuous functions with compact support on $\mathbb{R}$. The space $\mathscr{M}$ contains a cone of positive measures $\mathscr{M}_+$, i. e. of measures taking positive values on positive functions. One has the following inclusions: $\mathscr{S}' \subset \mathscr{D}'$ and $\mathscr{M}_+ \subset \mathscr{M} \subset \mathscr{D}'$. The intersection $\mathscr{M}_{sl} = \mathscr{M} \cap \mathscr{S}'$ is called the space of measures of slow growth. A measure μ of slow growth can be characterized by the property that for some positive integer k the integral

$$\int_{-\infty}^{+\infty} (x^2+1)^{-k} d\mu$$

converges (see [7, Thm. VII of Ch. VII]). In particular, from this criterion and the fact that the series $\sum\limits_{\rho \neq 0,1} |\rho|^{-2}$ converges ([3, Lemma 5.5]), we see that Δ_f is a measure of slow growth for any f.

Finally, we note that the Fourier transform $\hat{}$ is defined on $\mathscr{S}$ and $\mathscr{S}'$ and is a topological automorphism on these spaces. $\mathscr{D}$ is known to be dense in $\mathscr{S}$ and so $\hat{\mathscr{D}}$ is also dense in $\mathscr{S} = \hat{\mathscr{S}}$. To check that μ is a measure of slow growth it is enough to check that it is defined on a dense subset and that it is continuous on this dense subset in the topology of $\mathscr{S}$. In the same way, to check that a sequence of measures of slow growth converges to a measure of slow growth it is enough to check its convergence on a dense subset to a measure continuous on this dense subset. This follows from the definition of measures as linear functionals.

Our main tool will be a version of Weil explicit formula for L-functions of modular forms proven in [6, I.2] or in [3, Theorem 5.12] (in the last source some extra conditions on test functions are imposed).

Suppose $F \in \mathscr{S}(\mathbb{R})$ satisfies for some $\varepsilon > 0$ the following condition

$$|F(x)|, |F'(x)| \ll ce^{\left(-\frac{1}{2}+\varepsilon\right)|x|} \text{ as } |x| \to \infty. \tag{2.1}$$

Let

$$\Phi(s) := \int_0^\infty F(x) e^{\left(s-\frac{1}{2}\right)x} dx = \hat{F}(t),$$

where $s = \dfrac{1}{2} + it$. The next proposition gives us the explicit formula that we need to relate the sum over zeroes to the sum of coefficient of modular forms:

Proposition 2.1. *Let $f(z)$ be a primitive form of level N and weight k. Then the limit*

$$\sum_{L_f(\rho)=0} \Phi(\rho) = \lim_{T\to\infty} \sum_{\substack{L_f(\rho)=0 \\ |\rho|<T}} \Phi(\rho)$$

exists and we have the following formula:

$$\sum_{L_f(\rho)=0} \Phi(\rho) = -\sum_{p,m} b(p^m)(F(m\log p)+F(-m\log p))\frac{\log p}{p^{m/2}}+$$

$$+F(0)(\log N - 2\log(2\pi)) + \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\Phi\left(\frac{1}{2}+it\right)+\Phi\left(\frac{1}{2}-it\right)}{2}\cdot\psi\left(\frac{k}{2}+it\right)dt,$$

where $\psi(s)=\Gamma'(s)/\Gamma(s)$, $b(p^m)=(a_p)^m$ if $p|N$ and $b(p^m)=(\alpha_p)^m+(\bar{\alpha}_p)^m$ otherwise.

Taking a subsequence of $\{f_j\}$ we can assume that the limit

$$\alpha = \lim_{j\to\infty} \frac{\log N_j}{\log N_j + \log k_j}$$

exists. We will check the convergence of measures on $\hat{\mathscr{D}}$. From the above discussion this is enough to prove the result. Let us take any $\phi\in\hat{\mathscr{D}}$, $\phi=\hat{F}$, $F\in\mathscr{D}$. We have $\phi(t)=\Phi\left(\frac{1}{2}+it\right)$. The function F satisfies the condition (2.1), so we can apply the explicit formula to it. We fix $\phi(t)$ and let vary f_j Then, we get the equality when $j\to\infty$.

$$\Delta(\phi) = 2\pi F(0)\alpha + 2\int_{-\infty}^{+\infty} \frac{\phi(t)+\phi(-t)}{2}\cdot\lim_{j\to\infty}\frac{\psi\left(\frac{k_j}{2}+it\right)}{\log N_j+\log k_j}dt, \tag{2.2}$$

since $|b(p^m)|\leqslant 2$ and the integral is uniformly convergent as $\phi(t)\in\mathscr{S}$. The limit under the integral sign can be evaluated using the Stirling formula $\psi(s)=\log s + O\left(\frac{1}{|s|}\right)$ (see [5, p. 332]). This gives us

$$\lim_{j\to\infty}\frac{\psi\left(\frac{k_j}{2}+it\right)}{\log N_j+\log k_j} = \frac{1}{2}(1-\alpha).$$

But $\int_{-\infty}^{+\infty}\psi(t)dt = 2\pi F(0)$ and so the right hand side of (2.2) equals

$$2\pi F(0)\alpha + 2\pi F(0)(1-\alpha) = 2\pi F(0) = \int_{-\infty}^{+\infty}\phi(t)dt.$$

This concludes the proof of the theorem. □

Corollary 2.2. *Any fixed interval around* $s = \frac{1}{2}$ *contains zeroes of* $L_f(s)$ *if* q_f *is sufficiently large.*

Remark 2.3. One can prove a similar equidistribution statement for L-functions of bounded degree in the Selberg class, assuming suitable conjectures (like the Generalized Riemann Hypothesis). It is an interesting question how zeroes of L-functions are distributed if the degree of these L-functions grows with the analytic conductor. Some examples of non-trivial distributions of zeroes for Dedekind zeta functions are considered in [8].

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On the number of rational points of Jacobians over finite fields

(whith P. Lebacque)

Abstract. We prove lower and upper bounds for the class numbers of algebraic curves defined over finite fields. These bounds turn out to be better than most of the previously known bounds obtained using combinatorics. The methods used in the proof are essentially those from the explicit asymptotic theory of global fields. We thus provide a concrete application of effective results from the asymptotic theory of global fields and their zeta functions.

1. Introduction

1.1. Notation

We introduce the following notation:

X	a smooth projective absolutely irreducible curve over $\mathbb{F}_q$,
g	the genus of X,
K	the function field of X,
Φ_{q^f} or B_f	the number of places of K of degree f,
h	the class number of X (the number of $\mathbb{F}_q$-points of $\mathrm{Jac}(X)$),
$Z_X(T)$	the zeta function of X which is a rational function of T,
$\omega_i\sqrt{q}$	the inverse roots of the numerator of $Z_X(T)$,
κ	the residue of $Z_X(q^{-s})=\zeta_X(s)$ at $s=1$,
log	the Neperian logarithm $\log_e$.

By a *curve* we always mean a smooth projective absolutely irreducible curve.

1.2. Existing lower bounds for the class number

Our goal is to provide estimates for the number of rational points on the Jacobian of a smooth projective curve that use the information on the number of points on this curve defined over $\mathbb{F}_q$ or over its extensions. The

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starting point for all such estimates is the interpretation of the class number as the value at 1 of the numerator of the zeta function of the curve. In order to estimate it, one uses properties of the zeta function such as its functional equation, and the Riemann Hypothesis (Weil bounds).

From the work of Weil, we know that the class number h of a smooth projective absolutely irreducible curve X of genus g defined over $\mathbb{F}_q$ is bounded as follows:

$$(\sqrt{q}-1)^{2g} \leqslant h \leqslant (\sqrt{q}+1)^{2g}.$$

Considerable effort has been devoted to sharpening these bounds. Let us cite some work in this direction. Lachaud and Martin-Deschamps [5] first obtained the lower bound

$$h \geqslant h_{\mathrm{LMD}} = q^{g-1}\frac{(q-1)^2}{(q+1)(g+1)},$$

using a formula which is a consequence of the functional equation for the zeta function:

$$h = \frac{\sum\limits_{n=0}^{g-1} A_n + \sum\limits_{n=0}^{g-2} q^{g-1-n} A_n}{\sum\limits_{i=1}^{g} |1-\omega_i\sqrt{q}|^2},$$

where A_n is the number of effective divisors of degree n on X. Ever since, methods from combinatorics were used to give good bounds for the numerator and the denominator of this fraction.

In [2], [3], Ballet, Rolland, and Tutdere used this approach in order to prove rather elaborate lower bounds on h. Some of these bounds turn out to be asymptotically optimal when $g \to \infty$, meaning that they converge to the lower bound from the generalized Brauer—Siegel theorem for function fields ([9], see also Remark 2.8). The best of their lower bounds is given by the following theorem:

Theorem 1.1 (Ballet—Rolland—Tutdere). *Let $X/\mathbb{F}_q$ be a curve defined over $\mathbb{F}_q$ of genus $g \geqslant 2$ and of class number h. Let D_1, D_2 be finite sets of integers, $(\ell_r)_{r\in D_1}$, $(m_r)_{r\in D_2}$ be families of integers such that*:

1) $D_1 \subseteq \{1, \dots, g-1\}$;
2) $D_2 \subseteq \{1, \dots, g-2\}$;
3) *for any* $r \in D_1$, $\Phi_{q^r} \geqslant 1$;
4) *for any* $r \in D_2$, $\Phi_{q^r} \geqslant 1$;
5) $l_r \geqslant 0$ *and* $\sum\limits_{r\in D_1} r\ell_r \leqslant g-1$;
6) $m_r \geqslant 0$ *and* $\sum\limits_{r\in D_2} rm_r \leqslant g-2$.

Then $h \geqslant h_{\mathrm{BRT}}$ with

$$h_{\mathrm{BRT}} = \frac{(q-1)^2}{(g+1)(q+1)-\Phi_q}\left(\prod_{r\in D_1}\binom{\Phi_{q^r}+\ell_r}{\ell_r}+\right.$$

$$\left.+q^g\prod_{r\in D_2}\left[\left(\frac{q^r}{q^r-1}\right)^{\phi_{q^r}}-\Phi_{q^r}\binom{\Phi_{q^r}+m_r}{m_r}\int\limits_0^{q^{-r}}\frac{(q^{-r}-t)^{m_r}}{(1-t)^{\Phi_{q^r}+m_r+1}}\,dt\right]\right).$$

From now on we denote by h_{BRT} the best possible lower bound from this theorem, that is, the one with an optimal choice of $D_1, D_2, (\ell_r)_{r\in D_1}$, and $(m_r)_{r\in D_2}$.

In a recent article dealing with estimates for the number of points on general abelian varieties, Aubry, Haloui, and Lauchaud [1] obtained certain lower bounds on class numbers that can be very sharp when the curve in question has many rational points compared to its genus. However, these bounds are all rather poor from the asymptotic point of view when $g \to \infty$. Let us recall their results concerning the Jacobian of curves.

Theorem 1.2 (Aubry—Haloui—Lachaud). *For a smooth absolutely irreducible projective curve X defined over $\mathbb{F}_q$ of genus $g \geqslant 2$ and of class number h we have*:

1) $h \geqslant M(q)^g\left(q+1+\dfrac{\Phi_q-(q+1)}{g}\right)^g$ *with*

$$M(q) = \frac{e\log x^{1/x-1}}{x^{1/x}-1}, \quad x = \left(\frac{\sqrt{q}+1}{\sqrt{q}-1}\right)^2.$$

2) $h \geqslant \dfrac{q-1}{q^g-1}\left[\dbinom{\Phi_q+2g-2}{2g-1}+\displaystyle\sum_{r=2}^{2g-1}\Phi_{q^r}\binom{\Phi_q+2g-2-r}{2g-1-i}\right].$

3) *If $\Phi_q \geqslant g(\sqrt{q}-1)+1$ then*

$$h \geqslant \binom{\Phi_q+g-1}{g} - q\binom{\Phi_q+g-3}{g-2}.$$

4) $h \geqslant \dfrac{(q-1)^2}{(g+1)(q+1)-\Phi_q}\left[\dbinom{\Phi_q+g-2}{g-2}+\displaystyle\sum_{r=0}^{g-1}q^{g-1-1}\binom{\Phi_q+r-1}{r}\right].$

We denote by h_{AHL} the best possible lower bound for h given by (1)—(4) of this theorem. We remark that the estimate (3) can be very sharp when g is small and Φ_q is large. We will come back to that in § 3.

1.3. The aim of this paper is to show how the Mertens theorem and the explicit Brauer—Siegel theorem lead to improvements of these bounds in many cases, most notably when g is large. This is done in § 2 (Corollary 2.5). To do so we use the asymptotic theory of global fields, and more precisely the technique of explicit formulae. The third section is devoted to numerical experiments. We compare the bounds in several examples provided by recursive asymptotically good towers of function fields. Finally, in the fourth section we discuss further research directions and open problems.

2. Explicit formulae and their link to class numbers

2.1. Explicit formulae

Our starting point is the Mertens theorem [6] for curves and its relation to the generalized Brauer—Siegel theorem. Our exposition differs slightly from [6]: we take the opportunity to sharpen (and sometimes correct) the corresponding bounds.

Let us recall Serre's explicit formulae from [8].

Theorem 2.1 (Explicit formula). *For any sequence (v_n) such that the radius of convergence ρ of the series $\sum v_n t^n$ is strictly positive, define $\psi_{m,v}(t)=\sum_{n=1}^{\infty} v_{mn}t^{mn}$, and $\psi_v(t)=\psi_{1,v}(t)$. Then for $t<q^{-1}\rho$, we have the explicit formula*

$$\sum_{f=1}^{\infty} f\Phi_{q^f}\psi_{f,v}(t) = \psi_v(t)+\psi_v(qt) - \sum_{j=1}^{2g}\psi_v(\sqrt{q}\,\omega_j t).$$

We choose $N\in\mathbb{N}$, and take $v_n=1/n$ if $n\leqslant N$ and 0 otherwise. Applying Theorem 2.1 with $t=q^{-1}$, we obtain the identity

$$S_0(N) = S_1(N)+S_2(N)+S_3(N),$$

where

$$S_0(N) = \sum_{n=1}^{N} n^{-1}q^{-n}\sum_{m|n} m\Phi_{q^m} = \sum_{f=1}^{N}\frac{1}{fq^f}|X(\mathbb{F}_{q^f})|,$$

$$S_1(N) = \sum_{n=1}^{N}\frac{1}{n}, \qquad S_2(N) = \sum_{n=1}^{N}\frac{1}{nq^n}, \qquad S_3(N) = -\sum_{j=1}^{2g}\sum_{n=1}^{N}\frac{1}{n}(q^{-1/2}\omega_j)^n.$$

We transform it in order to make the desired quantities appear. For any $N \geqslant 1$,

$$\underbrace{S_0 - \sum_{f=1}^{N} \Phi_{q^f} \log\left(\frac{q^f}{q^f-1}\right)}_{\varepsilon_0(N)} + \sum_{f=1}^{N} \Phi_{q^f} \log\left(\frac{q^f}{q^f-1}\right) =$$

$$= S_1 + \underbrace{S_2 - \log\frac{q}{q-1}}_{\varepsilon_2(N)} + \log\frac{q}{q-1} +$$

$$+ \underbrace{S_3 - \log(\kappa \log q) + \log\frac{q}{q-1}}_{\varepsilon_3(N)} + \log(\kappa\log q) - \log\frac{q}{q-1}.$$

To get bounds for h we will not need estimates on $\varepsilon_0(N)$ and $\varepsilon_2(N)$, but they are useful for proving the Mertens theorem recalled later.

Lemma 2.2. *We have the following bounds for $\varepsilon_i(N)$:*

$$-\frac{c_1(q)}{Nq^{N/2}} - \frac{c_2(q)g}{Nq^{3N/4}} \leqslant \varepsilon_0(N) \leqslant 0,$$

$$-\frac{1}{(q-1)(N+1)q^N} \leqslant \varepsilon_2(N) \leqslant 0, \qquad 0 \leqslant |\varepsilon_3(N)| \leqslant \frac{2g}{(\sqrt{q}-1)(N+1)q^{N/2}},$$

with

$$c_1(q) = \frac{2q(q+1)}{(q-1)^2} \leqslant 12 \quad \textit{and} \quad c_2(q) = \frac{2q}{q-1}\left(\frac{\sqrt{q}}{\sqrt{q}-1} + \frac{q^{3/2}}{q^{3/2}-1}\right) \leqslant 20.$$

Proof. The following inequalities hold for $|x| > 1$ and $N > 0$:

$$\left|\log\left(\frac{x}{x-1}\right) - \sum_{n=1}^{N} \frac{1}{nx^n}\right| = \left|\sum_{n=N+1}^{\infty} \frac{1}{nx^n}\right| \leqslant \frac{1}{(N+1)|x|^{N+1}} \sum_{n=0}^{\infty} \frac{1}{|x|^n} \leqslant$$

$$\leqslant \frac{1}{(N+1)|x|^N(|x|-1)}.$$

This implies the bounds for $\varepsilon_2(N)$.

The one for $\varepsilon_3(N)$ is derived from the classical formula [9, Corollary 3.1.13]

$$\log(\kappa\log q) - \log\frac{q}{q-1} = \sum_{i=1}^{2g} \log\left(1 - \frac{\omega_j}{\sqrt{q}}\right).$$

It gives

$$|\varepsilon_3(N)| = \left| -\sum_{j=1}^{2g}\sum_{n=1}^{N}\frac{1}{n}(q^{-1/2}\omega_j)^n - \log(\kappa\log q) + \log\frac{q}{q-1}\right| =$$

$$= \left|\sum_{j=1}^{2g}\left(-\log\left(1-\frac{\omega_j}{\sqrt{q}}\right) - \sum_{n=1}^{N}\frac{1}{n}\left(\frac{\omega_j}{\sqrt{q}}\right)^n\right)\right|,$$

and since $|\omega_j| = 1$, we have

$$|\varepsilon_3(N)| \leqslant \sum_{j=1}^{2g}\frac{1}{(N+1)\sqrt{q}^N|\sqrt{q}-\omega_j|} \leqslant \frac{2g}{(\sqrt{q}-1)(N+1)q^{N/2}}.$$

We finally estimate $\varepsilon_0(N)$ along the lines of [6, proof of Lemma 2]. We first transform the expression for S_0:

$$S_0(N) = \sum_{f=1}^{N} f\Phi_{q^f}\sum_{m=1}^{[N/f]} q^{-fm}(fm)^{-1} = \sum_{f=1}^{N}\Phi_{q^f}\sum_{m=1}^{[N/f]}\frac{1}{q^{fm}m}.$$

Thus,

$$\varepsilon_0(N) = S_0(N) - \sum_{f=1}^{N}\Phi_{q^f}\log\frac{q^f}{q^f-1} = -\sum_{f=1}^{N}\Phi_{q^f}\left(\log\frac{q^f}{q^f-1} - \sum_{m=1}^{[N/f]}\frac{1}{q^{fm}m}\right) =$$

$$= -\sum_{f=1}^{N}\Phi_{q^f}\sum_{m=[N/f]+1}^{\infty}\frac{1}{q^{fm}m}.$$

As $\frac{1}{m} \leqslant \frac{1}{[N/f]+1}$, we get

$$0 \leqslant -\varepsilon_0(N) \leqslant \sum_{f=1}^{N}\frac{\Phi_{q^f}}{([N/f]+1)q^{f[N/f]}(q^f-1)}.$$

To estimate Φ_{q^f} we use $\Phi_{q^f} \leqslant \frac{q^f+1+2gq^{f/2}}{f}$. Thus

$$0 \leqslant -\varepsilon_0(N) \leqslant \frac{1}{N}\sum_{f=1}^{N}\frac{q^f+1+2gq^{f/2}}{(q^f-1)q^{f[N/f]}}.$$

We split the last sum in two, using the fact that for $f > [N/2]$ we have $[N/f]=1$, and for $f \leqslant [N/2]$ we have $f[N/f] \geqslant N-f$:

$$\begin{aligned}
-\varepsilon_0(N) &\leqslant \frac{1}{N}\sum_{f=1}^{[N/2]} \frac{q^f+1+2gq^{f/2}}{q^{N-f}(q^f-1)} + \frac{1}{N}\sum_{f=[N/2]+1}^{N} \frac{q^f+1+2gq^{f/2}}{q^f(q^f-1)} \leqslant \\
&\leqslant \frac{1}{N}\left(\sum_{f=1}^{[N/2]} \frac{q^f+1}{q^f-1}q^{f-N} + \sum_{f=[N/2]+1}^{N} \frac{q^f+1}{q^f-1}q^{-f}\right) + \\
&\quad + \frac{2g}{N}\left(\sum_{f=1}^{[N/2]} \frac{q^f}{q^f-1}q^{f/2-N} + \sum_{f=[N/2]+1}^{N} \frac{q^f}{q^f-1}q^{-3f/2}\right) \leqslant \\
&\leqslant \frac{q+1}{(q-1)N}\left(\sum_{f=1}^{[N/2]} q^{f-N} + \sum_{f=[N/2]+1}^{N} q^{-f}\right) + \\
&\quad + \frac{2gq}{N(q-1)}\left(\sum_{f=1}^{[N/2]} q^{f/2-N} + \sum_{f=[N/2]+1}^{N} q^{-3f/2}\right) \leqslant \\
&\leqslant \frac{(q+1)(q^{-[N/2]-1}+q^{-N+[N/2]})}{(q-1)N(1-q^{-1})} + \frac{2gq}{N(q-1)}\left(\frac{q^{-N+[N/2]/2}}{1-q^{-1/2}} + \frac{q^{-3([N/2]+1)/2}}{1-q^{-3/2}}\right) \leqslant \\
&\leqslant \frac{2(q+1)q}{(q-1)^2}\cdot\frac{1}{Nq^{N/2}} + \frac{2q}{q-1}\left(\frac{\sqrt{q}}{\sqrt{q}-1} + \frac{q^{3/2}}{q^{3/2}-1}\right)\frac{g}{Nq^{-3N/4}}. \qquad \square
\end{aligned}$$

Remark 2.3. The bound for $\varepsilon_0(N)$ provides a correction to [6, Lemma 2], and the bound for $\varepsilon_3(N)$ corrects Lemma 5 there. It can be easily checked that these bounds are also valid in the more general situation of varieties over finite fields treated in [6].

2.2. Bounds for the class number

Using the calculations from the previous section and applying the class number formula

$$\kappa \log q = \frac{hq^{1-g}}{q-1},$$

we get the following theorem.

Theorem 2.4. *Let X be a smooth projective absolutely irreducible curve defined over $\mathbb{F}_q$ of class number h. Then h is given by the following formula valid for any $N \geqslant 1$:*

$$\log h = g\log q + \sum_{f=1}^{N} \frac{1}{fq^f}|X(\mathbb{F}_{q^f})| - \sum_{n=1}^{N} \frac{1+q^{-n}}{n} - \varepsilon_3(N),$$

or equivalently,

$$\log h = g\log q + \sum_{r=1}^{N}\left(\Phi_{q^r}\sum_{f=1}^{\lfloor N/r\rfloor}\frac{1}{fq^{rf}}\right) - \sum_{n=1}^{N}\frac{1+q^{-n}}{n} - \varepsilon_3(N),$$

where $\varepsilon_3(N)$ *satisfies* $|\varepsilon_3(N)| \leqslant \dfrac{2g}{(\sqrt{q}-1)(N+1)q^{N/2}}$.

Corollary 2.5 (Bounds for the class number). *The number of rational points h on the Jacobian of X satisfies $h_{\min}(N) \leqslant h \leqslant h_{\max}(N)$, where*

$$h_{\min}(N) = q^g \exp\left(\sum_{f=1}^{N}\frac{1}{fq^f}|X(\mathbb{F}_{q^f})| - \sum_{n=1}^{N}\frac{1+q^{-n}}{n} - \frac{2g}{(\sqrt{q}-1)(N+1)q^{N/2}}\right),$$

$$h_{\max}(N) = q^g \exp\left(\sum_{f=1}^{N}\frac{1}{fq^f}|X(\mathbb{F}_{q^f})| - \sum_{n=1}^{N}\frac{1+q^{-n}}{n} + \frac{2g}{(\sqrt{q}-1)(N+1)q^{N/2}}\right).$$

Remark 2.6. The knowledge of a given (small) number of Φ_{q^f}'s allows us, nevertheless, to apply Corollary 2.5 for any N. For example, in the case of lower bounds, one can bound from below the unknown Φ_{q^f} by 0, or by the quantities arising from the Weil bounds, depending on which one is better. We thus get a family of bounds parametrized by N, and we can choose the best one.

2.3. Mertens theorem and class numbers

Putting together estimates from Section 2.1, we find once again:

Theorem 2.7 (Mertens theorem [6]). *Let X be a smooth projective absolutely irreducible curve of genus g defined over $\mathbb{F}_q$. Then*

$$\sum_{f=1}^{N}\Phi_{q^f}\log\left(\frac{q^f}{q^f-1}\right) = \log(\kappa\log q) - \varepsilon_0(N) + \varepsilon_2(N) + \varepsilon_3(N) - \sum_{n=1}^{N}\frac{1}{n}.$$

For any $N \geqslant 1$, we can deduce from this a weaker form of our bound, which might be easier to compare to Ballet—Rolland—Tutdere's bound:

$$\log h = g\log q + \left[\sum_{f=1}^{N}\Phi_{q^f}\log\left(\frac{q^f}{q^f-1}\right)\right] - \sum_{n=1}^{N}\frac{1+q^{-n}}{n} + \varepsilon_0(N) - \varepsilon_3(N).$$

Remark 2.8. Theorem 2.7 implies that our bounds on h are asymptotically optimal. More precisely, recall that a family of curves $\{X_i\}$ over $\mathbb{F}_q$ of genus $g_i \to \infty$ is *asymptotically exact* if the limits

$$\phi_{q^r} = \lim_{i\to\infty}\frac{\Phi_{q^r}(X_i)}{g_i}$$

exist for all r. For asymptotically exact families of curves the generalized Brauer—Siegel theorem [9] states that

$$\lim_{i\to\infty}\frac{\log h(X_i)}{g_i} = \log q + \sum_{r=1}^{\infty}\phi_{q^r}\log\left(\frac{q^r}{q^r-1}\right).$$

We see that when $g_i \to \infty$ and then $N \to \infty$, the bounds $h_{\min}(N)$ and $h_{\max}(N)$ from Corollary 2.5 divided by g_i converge to the right hand side of the above equality.

3. Numerical computations

In this section, we compare the lower bound $h_{\min}(N)$ given by Theorem 2.4 with h_{BRT} and h_{AHL} in the situation of recursive towers. We denote by h_{LZ} the bound from Theorem 2.4 for the optimal choice of N. Such a number N is found by computer-aided calculations where the missing information on the number of points on a curve X over $\mathbb{F}_{q^r}$ is obtained either from the inequality $X(\mathbb{F}_{q^r}) \geqslant X(\mathbb{F}_{q^d})$ when $d\,|\,r$, or from Serre's bound $X(\mathbb{F}_{q^r}) \geqslant q^r + 1 - g\lfloor 2q^{r/2}\rfloor$, depending on which one is more precise. We follow closely [3, Section 5].

Recall that a *tower* of function fields over $\mathbb{F}_q$ is an infinite sequence $\{F_k/\mathbb{F}_q\}_{k\in\mathbb{N}}$ of function fields such that for all k the ground field $\mathbb{F}_q$ is algebraically closed in F_k, $F_k \subset F_{k+1}$, and the genus satisfies $g(F_k) \to \infty$. A *recursive tower* is a tower $\{F_k\}$ of function fields over $\mathbb{F}_q$ such that $F_0 = \mathbb{F}_q(x_0)$ is a rational function field and $F_{k+1} = F_k(x_{k+1})$ where x_{k+1} satisfy the equation $f(x_k, x_{k+1}) = 0$ for a given polynomial $f(X,Y)$ in $\mathbb{F}_q[X,Y]$.

3.1. The first tower of Garcia—Stichtenoth

Assume that q^r is a square, and consider the tower $\{H_k\} = \mathscr{H}/\mathbb{F}_{q^r}$ defined recursively by the polynomial

$$f(X,Y) = Y^{q^{r/2}}X^{q^{r/2-1}} + Y - X^{q^{r/2}} \in \mathbb{F}_q[X,Y].$$

We also consider the recursive tower $\{F_k\} = \mathscr{F}/\mathbb{F}_q$ of function fields defined by the same polynomial starting with the rational function field $\mathbb{F}_q(x_0)$. The base change of F_k to $\mathbb{F}_{q^r}$ gives H_k.

We compare the numerical estimates from [3, Section 5.1] with what we obtain using our bound h_{LZ}. We take $q = 2$, $r = 2$ and consider the fields H_2, H_3, and H_4. Note an error in [3, Section 5.1] where for $k = 3$ the genus is erroneously taken to be equal to 14 instead of 13 (this was pointed out by Julia Pieltant). Recall that $B_1(H_k)$ denotes the number of $\mathbb{F}_4$-points of the curve corresponding to H_k.

Step k	$g(H_k)$	$B_1(H_k)$	h_{BRT}	h_{AHL}	h_{LZ}	N
2	5	16	7434	12240	9230	10
3	13	30	16 911 279 581	16 271 525 520	26 274 427 880	33
4	33	56	1.43×10^{25}	0.075×10^{25}	4.149×10^{25}	83

Here is a similar comparison for $q = 2$ and the tower $\mathscr{F}$ with $B_1(F_k)$ and $B_2(F_k)$ denoting respectively the number of $\mathbb{F}_2$- and $\mathbb{F}_4$-rational points of the curve corresponding to F_k:

Step k	$g(F_k)$	$B_1(F_k)$	$B_2(F_k)$	h_{BRT}	h_{LZ}	N
2	5	2	7	7	30	12
3	13	2	14	10453	42898	26
4	33	2	27	343 733 443 618	1 543 267 494 985	74

We notice that our bound is better than the other ones except for the case of $H_2/\mathbb{F}_4$ where we cannot beat h_{AHL}. The situation changes, however, if we use some additional information on the places of $H_2/\mathbb{F}_4$. Namely, one can calculate that $B_2(H_2) = 0$ and $B_3(H_2) = 24$. These values give the bound $h_{\mathrm{LZ}} = 13\,430$ reached for $N = 11$. Using MAGMA we calculated that the exact value of the class number is 16200.

3.2. The tower of Bassa—Garcia—Stichtenoth

Consider the tower $\{H_k\} = \mathscr{H}/\mathbb{F}_{q^3}$ defined recursively by the polynomial

$$f(X,Y) = (Y^q - Y)^{q-1} + 1 + \frac{X^{q(q-1)}}{(X^{q-1} - 1)^{q-1}} \in \mathbb{F}_q[X,Y],$$

and let $\{F_k\} = \mathscr{F}/\mathbb{F}_q$ be the same recursive tower over $\mathbb{F}_q$. We have the following numerical estimates for the class numbers when $q = 2$, that is, over $\mathbb{F}_8$ for H_k and over $\mathbb{F}_2$ for F_k. The value of h_{BRT} bound is taken from [3, Section 5.1].

Step k	$g(H_k)$	$B_1(H_k)$	h_{BRT}	h_{LZ}	N
2	5	24	125 537	126 832	9
3	13	48	2.556×10^{13}	4.039×10^{13}	29
4	29	96	2.010×10^{30}	5.778×10^{30}	11

Step k	$g(F_k)$	$B_3(F_k)$	h_{BRT}	h_{LZ}	N
2	5	8	3	3	5
3	13	16	771	1623	19
4	29	32	212 127 395	751 622 136	61

3.3. Composite towers

The next example is the composite tower $\{E_k/\mathbb{F}_{q^2}\}$ constructed in [4]. It is obtained as a composite of the tower of Garcia and Stichtenoth from Section 3.1 with a certain explicitly given function field. The details can be found in [3, Proposition 5.11]. The following table combines the estimates for $q^2 = 4$:

Step k	$g(E_k)$	$B_1(E_k)$	$B_2(E_k)$	$B_3(E_k)$	h_{BRT}	h_{LZ}	N
2	55	1	12	12	3.657×10^{31}	23.55×10^{31}	14
3	132	1	24	24	9.198×10^{77}	121.02×10^{77}	15

For two other composite towers $\{E_k/\mathbb{F}_2\}$ and $\{E'_k/\mathbb{F}_8\}$ this time based on the tower from Section 3.2 (see [3, Proposition 5.17] for a detailed description), we get the following numerical data:

Step k	$g(E_k)$	$B_3(E_k)$	$B_6(E_k)$	h_{BRT}	h_{LZ}	N
2	17	16	8	10 254	27563	30
3	49	32	16	1.718×10^{14}	9.173×10^{14}	94

Step k	$g(E'_k)$	$B_1(E'_k)$	$B_2(E'_k)$	h_{BRT}	h_{LZ}	N
2	17	48	24	1.002×10^{17}	2.304×10^{17}	35
3	49	96	48	2.426×10^{48}	13.08×10^{48}	10

One more composite tower $E_k/\mathbb{F}_4$ introduced in [10] (see also [3, Proposition 5.18]) gives us the following table:

Step k	$g(E_k)$	$B_1(E_k)$	$B_2(E_k)$	$B_3(E_k)$	h_{BRT}	h_{LZ}	N
2	30	1	9	9	4.625×10^{16}	18.329×10^{16}	52
3	89	1	27	27	2.236×10^{52}	21.39×10^{52}	16

For the composite tower $E_k/\mathbb{F}_9$ from [3, Proposition 5.20] we obtain:

Step k	$g(E_k)$	$B_1(E_k)$	$B_2(E_k)$	h_{BRT}	h_{LZ}	N
2	15	36	4	8.563×10^{14}	18.76×10^{14}	30
3	46	72	8	7.470×10^{45}	41.64×10^{45}	10

Finally, for yet another composite tower $E_k/\mathbb{F}_4$ from [3, Proposition 5.22] we get:

Step k	$g(E_k)$	$B_1(E_k)$	$B_2(E_k)$	h_{BRT}	h_{LZ}	N
2	25	36	9	1.415×10^{18}	3.835×10^{18}	56
3	124	108	27	3.501×10^{86}	36.23×10^{86}	16

In all these examples with one exception we manage to improve on the previously known bounds.

4. Open questions

Several natural questions arise in connection with the bounds obtained in this paper.

Question 4.1. Is it possible to compare the bounds h_{BRT}, h_{AHL}, and h_{LZ}?

We would like to have a more or less explicit description of the cases when each of the bounds is the best one. In the above examples our bound h_{LZ} always turned out to be better than h_{BRT}. However, we were not able to establish this fact in general. Comparing the bounds h_{LZ} and h_{BRT} does not seem to be easy, in particular due to the fact that the number N corresponding to the optimal $h_{\min}(N)$ can vary significantly and does not correspond at all to the number of known Φ_{q^r}'s.

Question 4.2. Can one improve (or even optimize) the bound h_{LZ} using different test functions in the explicit formulae?

Oesterlé managed to get the best possible bounds for $|X(\mathbb{F}_{q^r})|$ available from explicit formulae using the linear programming approach (see [8]). This technique, however, does not seem to be applicable directly in our case due to the non-linearity of the problem in question. The optimization seeming difficult, it would be interesting at least to find examples where a different choice of test functions in the explicit formulae leads to better bounds than h_{LZ}.

Question 4.3. What are the analogues of the above bounds in the number field case?

This question seems to be more directly accessible than the previous ones, since there are both the Mertens theorem and an explicit version of the Brauer—Siegel theorem available in the number field case [6], [7]. Nevertheless, analytic components of the proofs will certainly be more substantial, and the application of the Generalized Riemann Hypothesis might be necessary in certain cases.

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On M-functions associated with modular forms

(with P. Lebacque)

Abstract. Let f be a primitive cusp form of weight k and level N, let χ be a Dirichlet character of conductor coprime with N, and let $\mathfrak{L}(f\otimes\chi,s)$ denote either $\log L(f\otimes\chi,s)$ or $(L'/L)(f\otimes\chi,s)$. In this article we study the distribution of the values of $\mathfrak{L}$ when either χ or f vary. First, for a quasi-character $\psi\colon\mathbb{C}\to\mathbb{C}^\times$ we find the limit for the average $\mathrm{Avg}_\chi\psi(L(f\otimes\chi,s))$, when f is fixed and χ varies through the set of characters with prime conductor that tends to infinity. Second, we prove an equidistribution result for the values of $\mathfrak{L}(f\otimes\chi,s)$ by establishing analytic properties of the above limit function. Third, we study the limit of the harmonic average $\mathrm{Avg}^h_f\psi(L(f,s))$, when f runs through the set of primitive cusp forms of given weight k and level $N\to\infty$. Most of the results are obtained conditionally on the Generalized Riemann Hypothesis for $L(f\otimes\chi,s)$.

1. Introduction

1.1. Some history

The study of the distribution of values of L-functions is a classical topic in number theory. In the first half of 20th century Bohr, Jessen, Wintner, etc. intiated a study of the distribution of the values of the logarithm $\log\zeta(s)$ and the logarithmic derivative $(\zeta'/\zeta)(s)$ of the Riemann zeta function, when $\operatorname{Re}s=\sigma>\dfrac{1}{2}$ is fixed and $\operatorname{Im}s=\tau\in\mathbb{R}$ varies [1, 2, 18, 19]. This was later generalized to L-functions of cusp forms and Dedekind zeta functions by Matsumoto [23, 24, 25].

In the last decade Y. Ihara in [5] proposed a novel view on the problem by studying other families of L-functions. His initial motivation was to investigate the properties of the Euler—Kronecker constant γ_K of a global field K, which was defined by him in [4] to be the constant term of the Laurent series expansion of the logarithmic derivative of the Dedekind zeta function of K, $\zeta'_K(s)/\zeta_K(s)$. The study of $L'(1,\chi)/L(1,\chi)$

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initiated in [13] grew out to give a whole range of beautiful results on the value distribution of L'/L and $\log L$.

Given a global field K, i. e., a finite extension of $\mathbb{Q}$ or of $\mathbb{F}_q(t)$, and a family of characters χ of K Ihara considered in [5] the distribution of $L'(s,\chi)/L(s,\chi)$ in the following cases:

(A) K is $\mathbb{Q}$, a quadratic extension of $\mathbb{Q}$ or a function field over $\mathbb{F}_q$, and χ are Dirichlet characters on K;

(B) K is a number field with at least two archimedean primes, and χ are normalized unramified Grössencharacters;

(C) $K=\mathbb{Q}$ and $\chi=\chi_t$, $t\in\mathbb{R}$ defined by $\chi_t(p)=p^{-it}$.

The equidistribution results of the type

$$\operatorname{Avg}'_\chi \Phi\Big(\frac{L'(s,\chi)}{L(s,\chi)}\Big) = \int_{\mathbb{C}} M_\sigma(w)\Phi(w)|dw|, \tag{1}$$

(with a suitably defined average in each of the above cases) were proven for $\sigma=\operatorname{Re}s>1$ for number fields, and for $\sigma>3/4$ for function fields, under significant restrictions on the test function Φ. The function field case was treated once again in [9] by Y. Ihara and K. Matsumoto, with both the assumptions on Φ and on σ having been relaxed (Φ of at most polynomial growth and $\sigma>1/2$ respectively). The most general results in the direction of the case (A) were established in [11] conditionally under the Generalized Riemann Hypothesis (GRH) in the number field case and unconditionally in the function field case (the Weil's Riemann hypothesis being valid) for both families $L'(s,\chi)/L(s,\chi)$ and $\log L(s,\chi)$. For $\operatorname{Re}s>\frac{1}{2}$ Ihara and Matsumoto prove that

$$\operatorname{Avg}_\chi \Phi\Big(\frac{L'(s,\chi)}{L(s,\chi)}\Big) = \int_{\mathbb{C}} M_\sigma(w)\Phi(w)|dw|,$$

$$\operatorname{Avg}_\chi \Phi(\log L(s,\chi)) = \int_{\mathbb{C}} \mathscr{M}_\sigma(w)\Phi(w)|dw|,$$

for continuous test functions Φ of at most exponential growth. Note that Avg' in (1) is different from the one used in the latter paper, since extra averaging over conductors is assumed in the former case, the resulting statements being weaker.

Unconditional results for a more restrictive class of Φ (bounded continous functions), and with extra averaging over the conductor Avg', but still for $\operatorname{Re}s>\frac{1}{2}$ were established in [10] and [12] in the log and $\log'$ cases respectively in the situations (A, $K=\mathbb{Q}$) and (C).

The above results give rise to the density functions $M_\sigma(z)$ and a related function $\widetilde{M}_s(z_1, z_2)$ (which is the inverse Fourier transform of M_σ, when $z_2 = \bar{z}_1$, $s = \sigma \in \mathbb{R}$) both in the log and log$'$ cases. Under optimal circumstances (though it is very far from being known unconditionally in all cases) we have

$$M_\sigma(z) = \mathrm{Avg}_\chi\, \delta_z(\mathfrak{L}(\chi, s)), \quad \widetilde{M}_\sigma(z_1, z_2) = \mathrm{Avg}_\chi\, \psi_{z_1,z_2}(\mathfrak{L}(\chi, s)),$$

where $\mathfrak{L}(s, \chi)$ is either $L'(s, \chi)/L(s, \chi)$ or $\log L(s, \chi)$, δ_z is the Dirac delta function, and $\psi_{z_1,z_2}(w) = \exp\Big(\frac{i}{2}(z_1\bar{w} + z_2 w)\Big)$ is a quasi-character.

The functions M and $\widetilde{M}$ turn out to have some remarkable properties that can be established unconditionally. For example, $\widetilde{M}$ has an Euler product expansion, an analytic continuation to the left of $\mathrm{Re}\, s > 1/2$, its zeroes and the "Plancherel volume" $\int\limits_{\mathbb{C}} |\widetilde{M}_\sigma(z, \bar{z})|^2 |dz|$ are interesting objects to investigate. We refer to [6, 7] for an in-depth study of M and $\widetilde{M}$, as well as to the survey [8] for a thorough discussion of the above topics.

In a recent paper by M. Mourtada and K. Murty [27] averages over quadratic characters were considered. Using the methods from [11], they establish an equidistribution result conditional on GRH. Note that in their case the values taken by the L-functions are real. In this respect the situation is similar to the one considered by us in Section 5 in case we assume that s is real.

Finally, let us quote a still more recent preprint by K. Matsumoto and Y. Umegaki [26] that treats similar questions for differences of logarithms of two symmetric power L-functions under the assumption of the GRH. Their approach is based on [10] rather than on [11], though the employed techniques are remarkably close to the ones we apply in § 5. The results of Matsumoto and Umegaki are complementary to ours, since the case of $\mathrm{Sym}^1 f = f$, which is the main subject of our paper, could not be treated in [26].

1.2. Main results

In this article, we generalize to the case of modular forms the methods of Ihara and Matsumoto to understand the average values of L-functions of Dirichlet characters over global fields.

Our results are obtained in two different settings. First, we consider the case of a fixed modular form, while averaging with respect to its twists by Dirichlet characters. Our results in this setting are fairly complete, though sometimes conditional on GRH. Second, we consider averages with respect to primitive forms of given weight and level, when the level goes to infinity.

Let us formulate our main results. A more thorough presentation of the corresponding notation can found in Section 2 and in the corresponding sections.

Let $B_k(N)$ denote the set of primitive cusp forms of weight k and level N, let $f \in B_k(N)$, and let χ be a Dirichlet character of conductor m coprime with N. Define $\mathfrak{L}(f\otimes\chi, s)$ to be either $(L'/L)(f\otimes\chi, s)$ or $\log L(f\otimes\chi, s)$, put $\mathfrak{g}(f\otimes\chi, s, z) = \exp\Big(\frac{iz}{2}\mathfrak{L}(f\otimes\chi, s)\Big)$. We introduce $\mathfrak{l}_z(n)$ to be the coefficients of the Dirichlet series expansion $\mathfrak{g}(f, s, z) :=$ $:= \exp\Big(\frac{iz}{2}\mathfrak{L}(f, s)\Big) = \sum\limits_{n\geqslant 1} \mathfrak{l}_z(n)n^{-s}$. Using the relations between the coefficients of the Dirichlet series expansion $L(f, s) = \sum\limits_{n\geqslant 1} \eta_f(n)n^{-s}$, one can write $\mathfrak{l}_z(n) = \sum\limits_{x\geqslant 1} c^N_{z,x}(n)\eta_f(x)$, where $c^N_{z,x}(n)$ depend only on the level N. Put $c_{z,x}(n) = c^1_{z,x}(n)$.

In what follows, the expressions of the form $f \ll g$, $g \gg f$, and $f = O(g)$ all denote that $|f| \leqslant c|g|$, where c is a positive constant. The dependence of the constant on additional parameters will be explicitly indicated (in the form $\ll_{\varepsilon,\delta,\dots}$ or $O_{\varepsilon,\delta,\dots}$), if it is not stipulated otherwise in the text. We denote by $v_p(n)$ the p-adic valuation of n, writing as well $p^k \,\|\, n$ if $v_p(n) = k$. We also use the notation := or =: meaning that the corresponding object to the left or to the right of the equality respectively is defined in this way.

Our main results are as follows.

Theorem (Theorem 3.1)**.** *Assume that m is a prime number and let Γ_m denote the group of Dirichlet characters modulo m. Let $0 < \varepsilon < \frac{1}{2}$ and $T, R > 0$. Let $s = \sigma + it$ belong to the domain $\sigma \geqslant \varepsilon + \frac{1}{2}$, $|t| \leqslant T$, let z and z' be inside the disk $\mathcal{D}_R = \{z\colon |z| \leqslant R\}$. Then, assuming the Generalized Riemann Hypothesis (GRH) for $L(f\otimes\chi, s)$, we have*

$$\lim_{m\to\infty} \frac{1}{|\Gamma_m|} \sum_{\chi\in\Gamma_m} \overline{\mathfrak{g}(f\otimes\chi, s, z)}\mathfrak{g}(f\otimes\chi, s, z') =$$
$$= \sum_{n\geqslant 1} \overline{\mathfrak{l}_z(n)}\mathfrak{l}_{z'}(n)n^{-2\sigma} =: \widetilde{M}_\sigma(-\bar{z}, z').$$

Moreover the convergence is uniform in z, z' and t in the prescribed range.

Theorem (Theorem 4.1)**.** *Let $\operatorname{Re} s = \sigma > \frac{1}{2}$ and let m run over prime numbers. Let Φ be either a continuous function on $\mathbb{C}$ with at most expo-*

nential growth, or the characteristic function of a bounded subset of $\mathbb{C}$ or of a complement of a bounded subset of $\mathbb{C}$. Define M_σ as the inverse Fourier transform of $\widetilde{M}_\sigma(z,\bar{z})$. Then under GRH for $L(f\otimes\chi,s)$ we have

$$\lim_{m\to\infty}\frac{1}{|\Gamma_m|}\sum_{\chi\in\Gamma_m}\Phi(\mathfrak{L}(f\otimes\chi,s))=\int_{\mathbb{C}}M_\sigma(w)\Phi(w)|dw|.$$

Theorem (Theorem 5.1). *Assume that N is a prime number and that k is fixed. Let $0<\varepsilon<\frac{1}{2}$ and $T,R>0$. Let $s=\sigma+it$ belong to the domain $\sigma\geqslant\varepsilon+\frac{1}{2}$, $|t|\leqslant T$, and z and z' to the disc $\mathscr{D}_R$ of radius R. Then, assuming GRH for $L(f,s)$, we have*

$$\lim_{N\to+\infty}\sum_{f\in B_k(N)}\omega(f)\overline{\mathfrak{g}(f,s,z)}\,\mathfrak{g}(f,s,z')=\sum_{n,m\in\mathbb{N}}n^{-\bar{s}}m^{-s}\sum_{x\geqslant1}\overline{c_{z,x}(n)}c_{z',x}(m),$$

where $\omega(f)$ are the harmonic weights defined in Section 5. *Moreover the convergence is uniform on z,z' and t in the prescribed range.*

Finally, let us describe the structure of the paper. In Section 2 we introduce the notation and some technical lemmas to be used throughout the paper. The Section 3 is devoted to the proof of Theorem 3.1 on the mean values of the logarithms and logarithmic derivatives of L-functions obtained by taking averages over the twists of a given primitive modular form. Using GRH, we deduce it from Ihara and Matsumoto's results. In Section 4 we study unconditionally the analytic properties of M and $\widetilde{M}$ functions in the above setting. We then prove an equidistribution result (Theorem 4.1), which is, once again, conditional on GRH. In Section 5, we consider the average over primitive forms of given weight k and level N, when $N\to\infty$, establishing under GRH Theorem 5.1. The orthogonality of characters is replaced by the Petersson formula in this case, which obviously makes the proofs trickier. Finally, open questions, remarks and further research directions are discussed in Section 6.

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2. Notation

The goal of this section is to introduce the notation necessary to state our main results. We also prove some auxiliary estimates to be used throughout the paper.

2.1. The $\mathfrak{g}$-functions

Let N, k be two integers. We denote by $S_k(N)$ the set of cusp forms of weight k and level N, and by $S_k^{\mathrm{new}}(N)$ the set of new forms. For $f \in S_k(N)$ we write $f(z) = \sum_{n=1}^{\infty} \eta_f(n) n^{(k-1)/2} e(nz)$ for its Fourier expansion at the cusp ∞, with the standard notation $e(nz) = e^{2\pi i nz}$.

Let $B_k(N)$ denote the set of primitive forms of weight k and level N, i. e., the set of $f^{\mathrm{nor}} = f/\eta_f(1)$ where f runs through an orthogonal basis of $S_k^{\mathrm{new}}(N)$ consisting of eigenvectors of all Hecke operators T_n, so that the Fourier coefficients of the elements of $B_k(N)$ are the same as their Hecke eigenvalues. Note that for a primitive form $f \in B_k(N)$ all its coefficients $\eta_f(n)$ are real.

The L-function of a primitive form $f \in B_k(N)$ is defined as the Dirichlet series $L(f,s) = \sum_{n=0}^{\infty} \eta_f(n) n^{-s}$. The series converges absolutely for $\operatorname{Re} s > 1$, however, $L(f,s)$ can be analytically continued to an entire function on $\mathbb{C}$. It admits the Euler product expansion:

$$L(f,s) = \prod_p L_p(f,s),$$

where, for any prime number p,

$$L_p(f,s) = \begin{cases} \left(1 - \eta_f(p)p^{-s} + p^{-2s}\right)^{-1} & \text{if } (p,N) = 1, \\ \left(1 - \eta_f(p)p^{-s}\right)^{-1} & \text{if } p \mid N. \end{cases}$$

Notice that in this paper all L-functions are normalized so that the functional equation relates the values at s and $1-s$. By the results of Deligne, these local factors can be written as follows ([14, Ch. 6] or [20, Ch. IX, § 7]):

$$L_p(f,s) = \left(1 - \alpha_f(p)p^{-s}\right)^{-1}\left(1 - \beta_f(p)p^{-s}\right)^{-1}, \tag{2}$$

where

$$\begin{cases} |\alpha_f(p)| = 1, \ \beta_f(p) = \alpha_f(p)^{-1} & \text{if } (p,N) = 1, \\ \alpha_f(p) = \pm p^{-\frac{1}{2}}, \beta_f(p) = 0 & \text{if } p \parallel N \text{ (that is } p \mid N \text{ and } p^2 \nmid N), \\ \alpha_f(p) = \beta_f(p) = 0 & \text{if } p^2 \mid N. \end{cases}$$

We are interested in the two functions

$$\begin{cases} \mathfrak{g}(f,s,z) = \exp\!\left(\frac{iz}{2}\frac{L'(f,s)}{L(f,s)}\right), \\ G(f,s,z) = \exp\!\left(\frac{iz}{2}\log L(f,s)\right). \end{cases}$$

Define $h_n(x)$ and $H_n(x)$ as the coefficients of the following generating functions:

$$\exp\Big(\frac{xt}{1-t}\Big) = \sum_{n=0}^{+\infty} h_n(x)t^n, \quad \exp(-x\log(1-t)) = \sum_{n=0}^{+\infty} H_n(x)t^n,$$

or, equivalently (cf. [11, § 1.2]), as the functions given by $h_0(x)=H_0(x)=1$ and, for $n\geqslant 1$,

$$h_n(x) = \sum_{r=0}^{n} \frac{1}{r!}\binom{n-1}{r-1}x^r, \quad H_n(x) = \frac{1}{n!}x(x+1)\dots(x+n-1).$$

As we have

$$\frac{iz}{2}\frac{L'(f,s)}{L(f,s)} = -\frac{iz}{2}\sum_p \frac{\alpha_f(p)p^{-s}\log p}{1-\alpha_f(p)p^{-s}} + \frac{\beta_f(p)p^{-s}\log p}{1-\beta_f(p)p^{-s}},$$

we can write (using the standard convention that, in the case when $\beta_f(p)=0$, we put $\beta_f(p)^n=0$, if $n>0$, and $\beta_f(p)^0=1$):

$$\begin{aligned}
g(f,s,z) &= \exp\Big(\frac{iz}{2}\frac{L'(f,s)}{L(f,s)}\Big) = \\
&= \prod_p \exp\Big(\frac{\alpha_f(p)p^{-s}}{1-\alpha_f(p)p^{-s}}\cdot\frac{-iz\log p}{2}\Big)\exp\Big(\frac{\beta_f(p)p^{-s}}{1-\beta_f(p)p^{-s}}\cdot\frac{-iz\log p}{2}\Big) = \\
&= \prod_p \Big(\sum_n h_n\Big(-\frac{iz}{2}\log p\Big)\alpha_f(p)^n p^{-ns}\Big)\Big(\sum_n h_n\Big(-\frac{iz}{2}\log p\Big)\beta_f(p)^n p^{-ns}\Big) = \\
&= \prod_p \Big(\sum_{n=0}^{+\infty}\sum_{r=0}^{n} h_r\Big(-\frac{iz}{2}\log p\Big)h_{n-r}\Big(-\frac{iz}{2}\log p\Big)\alpha_f(p)^r\beta_f(p)^{n-r}p^{-ns}\Big) = \\
&= \prod_{p\nmid N} \Big(\sum_{n=0}^{+\infty}\sum_{r=0}^{n} h_r\Big(-\frac{iz}{2}\log p\Big)h_{n-r}\Big(-\frac{iz}{2}\log p\Big)\alpha_f(p)^{2r-n}p^{-ns}\Big)\times \\
&\quad \times \prod_{p\| N}\sum_{n=0}^{+\infty} h_n\Big(-\frac{iz}{2}\log p\Big)\alpha_f(p)^n p^{-ns} =: \prod_p \sum_{n=0}^{+\infty}\lambda_z(p^n)p^{-ns}.
\end{aligned}$$

In a similar way we get:

$$\begin{aligned}
G(f,s,z) &= \exp\Big(\frac{iz}{2}\log L(f,s)\Big) = \\
&= \prod_p \exp\Big(-\frac{iz}{2}\log(1-\alpha_p(f)p^{-s})\Big)\exp\Big(-\frac{iz}{2}\log(1-\beta_p(f)p^{-s})\Big) = \\
&= \prod_{p\nmid N}\Big(\sum_{n=0}^{+\infty}\sum_{r=0}^{n} H_r\Big(\frac{iz}{2}\Big)H_{n-r}\Big(\frac{iz}{2}\Big)\alpha_f(p)^{2r-n}p^{-ns}\Big)\times \\
&\quad \times \prod_{p\| N}\sum_{n=0}^{+\infty} H_n\Big(\frac{iz}{2}\Big)\alpha_f(p)^n p^{-ns} =: \prod_p\sum_{n=0}^{+\infty}\Lambda_z(p^n)p^{-ns}.
\end{aligned}$$

We extend multiplicatively λ_z and Λ_z to $\mathbb{N}$ so that we can write:

$$g(f,s,z)=\sum_{n\geqslant 1}\lambda_z(n)n^{-s},\quad G(f,s,z)=\sum_{n\geqslant 1}\Lambda_z(n)n^{-s}.$$

We will use the notation $\mathfrak{L}$ for $\dfrac{L'(f,s)}{L(f,s)}$ or $\log L(f,s)$, $\mathfrak{g}$ for g or G, $\mathfrak{h}_z(p^n)$ for $h_n\Big(-\frac{iz}{2}\log p\Big)$ or $H_n\Big(\frac{iz}{2}\Big)$, and $\mathfrak{l}$ for λ or Λ depending on the case we consider. Thus, we can write in a uniform way:

$$\mathfrak{g}(f,s,z)=\exp\Big(\frac{iz}{2}\mathfrak{L}(f,s)\Big)=\sum_{n\geqslant 1}\mathfrak{l}_z(n)n^{-s}=\prod_p\sum_{n=0}^{+\infty}\mathfrak{l}_z(p^n)p^{-ns}=$$

$$=\prod_{p\nmid N}\Bigg(\sum_{n=0}^{+\infty}\sum_{r=0}^{n}\mathfrak{h}_z(p^r)\mathfrak{h}_z(p^{n-r})\alpha_f(p)^{2r-n}p^{-ns}\Bigg)\prod_{p\|N}\sum_{n=0}^{+\infty}\mathfrak{h}_z(p^n)\alpha_f(p)^n p^{-ns}.$$

The coefficients $\mathfrak{l}_z(n)$ will be used to define the $\widetilde{M}$-functions in the case of averages over twists of modular forms by Dirichlet characters.

2.2. The coefficients $\mathfrak{l}_z(n)$ and $c_{z,x}(n)$

In this subsection we will find a more explicit expression for $\mathfrak{l}_z(n)$. For $p\nmid N$ we will use the formula (see [31, (3.5)])

$$\eta_f(p^r)=\frac{\alpha_f(p)^{r+1}-\beta_f(p)^{r+1}}{\alpha_f(p)-\beta_f(p)},$$

which easily follows from (2). Taking into account that $\beta_f(p)=\bar{\alpha}_f(p)$, we have for $r\geqslant 2$

$$\begin{aligned}\eta_f(p^r)&=\frac{\alpha_f(p)^{r+1}-\overline{\alpha_f(p)}^{r+1}}{\alpha_f(p)-\overline{\alpha_f(p)}}=\sum_{i=0}^{r}\alpha_f(p)^i\overline{\alpha_f(p)}^{r-i}=\sum_{i=0}^{r}\alpha_f(p)^{r-2i}=\\&=\alpha_f(p)^r+\overline{\alpha_f(p)}^r+\sum_{i=1}^{r-1}\alpha_f(p)^{r-2i}=\\&=\alpha_f(p)^r+\overline{\alpha_f(p)}^r+\sum_{i=0}^{r-2}\alpha_f(p)^{r-2i-2}=\\&=\alpha_f(p)^r+\overline{\alpha_f(p)}^r+\eta_f(p^{r-2}).\end{aligned}$$

The above formula also holds for $r=1$ if we put $\eta_f(p^{-1})=0$. From this we deduce that

$$\alpha_f(p)^r+\beta_f(p)^r=\eta_f(p^r)-\eta_f(p^{r-2}).$$

Using the previous formula, we can write

$$\begin{aligned}
\mathfrak{l}_z(p^r) &= \sum_{a=0}^{r} \mathfrak{h}_z(p^a)\mathfrak{h}_z(p^{r-a})\alpha_f(p)^{2a-r}\\
&= \mathfrak{h}_z(p^{\frac{r}{2}})^2 + \sum_{a=0}^{\lfloor\frac{r-1}{2}\rfloor} \mathfrak{h}_z(p^a)\mathfrak{h}_z(p^{r-a})\big(\alpha_f(p)^{r-2a}+\alpha_f(p)^{2a-r}\big) =\\
&= \mathfrak{h}_z(p^{\frac{r}{2}})^2 + \sum_{a=0}^{\lfloor\frac{r-1}{2}\rfloor} \mathfrak{h}_z(p^a)\mathfrak{h}_z(p^{r-a})\big(\eta_f(p^{r-2a})-\eta_f(p^{r-2a-2})\big) =\\
&= \mathfrak{h}_z(p^{\frac{r}{2}})^2 - \mathfrak{h}_z(p^{\frac{r}{2}-1})\mathfrak{h}_z(p^{\frac{r}{2}+1}) +\\
&\qquad + \sum_{a=0}^{\lfloor\frac{r-1}{2}\rfloor} (\mathfrak{h}_z(p^a)\mathfrak{h}_z(p^{r-a}) - \mathfrak{h}_z(p^{a-1})\mathfrak{h}_z(p^{r-a+1}))\eta_f(p^{r-2a}) =\\
&= \sum_{a=0}^{\lfloor\frac{r}{2}\rfloor} (\mathfrak{h}_z(p^a)\mathfrak{h}_z(p^{r-a}) - \mathfrak{h}_z(p^{a-1})\mathfrak{h}_z(p^{r-a+1}))\eta_f(p^{r-2a}),
\end{aligned}$$

where we put $\mathfrak{h}_z(p^{\frac{r}{2}}) = \mathfrak{h}_z(p^{\frac{r}{2}-1}) = 0$, if r is odd, and $\mathfrak{h}_z(p^a) = 0$, if $a < 0$.

When $p \mid N$ we have

$$\mathfrak{l}_z(p^r) = \mathfrak{h}_z(p^r)\alpha_f(p)^r = \mathfrak{h}_z(p^r)\eta_f(p)^r = \mathfrak{h}_z(p^r)\eta_f(p^r).$$

Denoting by $\mathscr{P}$ the set of prime numbers, for $n = \prod_{p\in\mathscr{P}} p^{v_p(n)}$ put

$$I_N(n) = \{m \in \mathbb{N} \colon v_p(m) \equiv v_p(n) \bmod 2 \text{ for } p \in \mathscr{P}, v_p(n) = v_p(m) \text{ if } p \mid N\}$$

and

$$J_N(n) = \{m \in I_N(n) \colon v_p(m) \leqslant v_p(n) \text{ for all } p \in \mathscr{P}\}.$$

Note the following easy estimate [3, Theorem 315], in which $\tau(n)$ is the number of divisors of n:

$$|J_N(n)| = \prod_{p\mid n}\Big(\Big\lfloor \frac{v_p(n)}{2}\Big\rfloor + 1\Big) \leqslant \tau(n) \ll_\varepsilon n^\varepsilon. \tag{3}$$

The previous computations may be summarized as follows:

$$\mathfrak{l}_z(p^r) = \sum_{x\in J_N(p^r)} c^N_{z,x}(p^r)\eta_f(x),$$

where

$$c^N_{z,p^a}(p^r) = \\
= \begin{cases} \mathfrak{h}_z(p^{\frac{r-a}{2}})\mathfrak{h}_z(p^{\frac{r+a}{2}}) - \mathfrak{h}_z(p^{\frac{r-a}{2}-1})\mathfrak{h}_z(p^{\frac{r+a}{2}+1}), & \text{if } p \nmid N \text{ and } r \equiv a \bmod 2,\\ \mathfrak{h}_z(p^r), & \text{if } p \mid N \text{ and } r = a,\\ 0, & \text{otherwise.} \end{cases}$$

We have $\mathfrak{l}_z(n)=\prod_{p|n}\mathfrak{l}_z(p^{v_p(n)})$ and $\eta_f(n)\eta_f(m)=\eta_f(nm)$ if $(n,m)=1$, thus

$$\mathfrak{l}_z(n)=\prod_{p|n}\Big(\sum_{x\in J_N(p^{v_p(n)})}c^N_{z,x}(p^{v_p(n)})\eta_f(x)\Big)=\sum_{x\in J_N(n)}c^N_{z,x}(n)\eta_f(x),$$

with

$$c^N_{z,x}(n)=\prod_{p|n}c^N_{z,p^{v_p(x)}}(p^{v_p(n)}).$$

Note that the coefficients $c^N_{z,x}(n)$, $I_N(n)$, and $J_N(n)$ depend only on the level N and not directly on the modular form f. Let us also define $I(n)=I_1(n)$, $J(n)=J_1(n)$, and $c_{z,x}(n)=c^1_{z,x}(n)$. They will be employed in the statement of Theorem 5.1, which is our main result on averages over the set of primitive forms $B_k(N)$.

Let $B(a,R)=\{z\in\mathbb{C}\colon |z-a|<R\}$ denote the open disc of radius R and center $a\in\mathbb{C}$, let $\overline{B(a,R)}$ be the corresponding closed disc. We also put $\mathcal{D}_R=\overline{B(0,R)}$. The following estimate is used throughout the paper.

Lemma 2.1. *For any $\varepsilon>0$ and $z\in\mathcal{D}_R$ we have $|c^N_{z,x}(n)|\ll_{\varepsilon,R}n^{\varepsilon}$ and $|\mathfrak{l}_z(n)|\ll_{\varepsilon,R}n^{\varepsilon}$.*

Proof. To see this, recall [11, 3.1.2] that for any prime p

$$\Big|H_r\Big(\frac{iz}{2}\Big)\Big|\leqslant H_r\Big(\frac{|z|}{2}\Big)\leqslant h_r\Big(\frac{|z|}{2}\Big)\leqslant h_r(|z|\log p)$$

and

$$\Big|h_r\Big(-\frac{iz}{2}\log p\Big)\Big|\leqslant h_r(|z|\log p)\leqslant \exp\big(2\sqrt{r|z|\log p}\big),$$

thus in both cases $|\mathfrak{h}_z(p^r)|\leqslant\exp\big(2\sqrt{r|z|\log p}\big)$. Using the concavity of the function $\sqrt{x}$, we see that

$$\begin{aligned}|c^N_{z,x}(p^r)|&\leqslant e^{2\sqrt{\frac{r-a}{2}|z|\log p}}e^{2\sqrt{\frac{r+a}{2}|z|\log p}}+e^{2\sqrt{(\frac{r-a}{2}-1)|z|\log p}}e^{2\sqrt{(\frac{r+a}{2}+1)|z|\log p}}\leqslant\\ &\leqslant e^{2\sqrt{|z|\log p}\left(\sqrt{\frac{r-a}{2}}+\sqrt{\frac{r+a}{2}}\right)}+e^{2\sqrt{|z|\log p}\left(\sqrt{\frac{r-a}{2}-1}+\sqrt{\frac{r+a}{2}+1}\right)}\leqslant\\ &\leqslant e^{2\sqrt{|z|\log p}\sqrt{2r}}+e^{2\sqrt{|z|\log p}\sqrt{2r}}\leqslant 2e^{2\sqrt{2r|z|\log p}}.\end{aligned}$$

when $p\nmid N$. The above estimates on $\mathfrak{h}_z(p^r)$ also imply the same bound on $c^N_{z,x}(p^r)$ when $p\mid N$.

Now, denoting by $\omega(n)$ the number of distinct prime divisors of n and using once again the concavity of $\sqrt{x}$, for $n=\prod_{p\in\mathcal{P}} p^{v_p(n)}$ we have

$$\log|c^N_{z,x}(n)| \leqslant \sum_{p|n}(\log 2+\sqrt{v_p(n)\log p}\sqrt{8R}) \ll_R \Big(\sum_{p|n}\sqrt{v_p(n)\log p}\Big)\sqrt{8R} \ll$$
$$\ll_R \sqrt{\sum_{p|n} v_p(n)\log p}\sqrt{\omega(n)} \ll_R \sqrt{\frac{\log n}{2+\log\log n}}\sqrt{\log n},$$

since by [5, Sublemma 3.10.5] (which is classical in the case of $\mathbb{N}$) we have

$$\omega(n) \ll \frac{\log n}{2+\log\log n}. \tag{4}$$

We thus conclude that $|c^N_{z,x}(n)| \ll_{\varepsilon,R} n^{\varepsilon}$.

As for the second statement, we notice that the estimate (3) together with Deligne bound $|\eta_f(n)| \leqslant \tau(n) \ll_\varepsilon n^\varepsilon$ imply

$$\iota_z(n) \ll_\varepsilon |J_N(n)|\cdot n^{\varepsilon}\cdot\tau(n) \ll_\varepsilon n^{3\varepsilon}.$$

□

We conclude the section by the following trivial but useful lemma.

Lemma 2.2. *We have* $\overline{\iota_z(n)} = \iota_{-\bar{z}}(n)$, *and* $\overline{c^N_{z,x}(n)} = c^N_{-\bar{z},x}(n)$.

Proof. The eigenvalues $\eta_f(n)$ are all real, so the L-functions $L(f,s)$ have Dirichlet series with real coefficients. Thus the statement of the lemma follows from the definition of the coefficients $\iota_z(n)$, and $c^N_{z,x}(n)$. □

3. Average on Twists

This section is devoted to the proof of an averaging result for twists of a given primitive form. It is to a large extent based on the work of Ihara and Matsumoto [11], which provides a general setting for the problem we consider.

3.1. Setting

Let us fix a primitive cusp form $f\in B_k(N)$ of weight k and level N. Let $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ be a primitive character mod m, where $(m,N)=1$. It is known (see [15, Prop. 14.19 and Prop. 14.20]) that $f\otimes\chi$ is a primitive form of weight k, level Nm^2, and nebentypus χ^2. We consider the twisted L-function given by

$$L(f\otimes\chi,s) = \prod_p L_p(f\otimes\chi,s),$$

where the local factors are defined as follows:

$$L_p(f\otimes\chi,s)=\big(1-\alpha_f(p)\chi(p)p^{-s}\big)^{-1}\big(1-\beta_f(p)\chi(p)p^{-s}\big)^{-1},$$

with the notation of Section 2. It is an L-function of degree 2 and conductor Nm^2, entire and polynomially bounded in vertical strips. After multiplication by the gamma factor

$$\gamma_k(s)=\sqrt{\pi}\,2^{\frac{3-k}{2}}(2\pi)^{-s}\Gamma\Big(s+\frac{k-1}{2}\Big),$$

it satisfies a functional equation [15, § 5.11]. Its analytic conductor $\mathfrak{q}(f\otimes\chi,s)$ is defined as follows:

$$\mathfrak{q}(f\otimes\chi,s)=Nm^2\Big(\Big|s+\frac{k-1}{2}\Big|+3\Big)\Big(\Big|s+\frac{k+1}{2}\Big|+3\Big)\leqslant Nm^2(|s|+k+3)^2.$$

Just as in Section 2 we use the following notation for modular forms with nebentypus:

$$\begin{cases} g(f\otimes\chi,s,z)=\exp\Big(\dfrac{iz}{2}\dfrac{L'(f\otimes\chi,s)}{L(f\otimes\chi,s)}\Big),\\ G(f\otimes\chi,s,z)=\exp\Big(\dfrac{iz}{2}\log L(f\otimes\chi,s)\Big).\end{cases}$$

We also write $\mathfrak{g}(f\otimes\chi,s,z)$ to denote either of the above two functions.

If G is a function on a finite group K, let $\operatorname{Avg}_{\chi\in K}G(\chi)$ denote the usual average $|K|^{-1}\sum_{\chi\in K}G(\chi)$.

3.2. The $\widetilde{M}$-function

We would like to understand the average over all Dirichlet characters mod m of the functions $\mathfrak{g}(f\otimes\chi,s,z)$, when m runs through large prime numbers. Ihara and Matsumoto's results apply in this case and we get the following theorem.

Theorem 3.1. *Assume that m is a prime number. Let Γ_m denote the group of Dirichlet characters modulo m. Let $0<\varepsilon<\frac{1}{2}$ and $T,R>0$. Let $s=\sigma+it$ belong to the domain $\sigma\geqslant\varepsilon+\frac{1}{2}$, $|t|\leqslant T$, let z and z' be inside the disk $\mathscr{D}_R$. Then, assuming the Generalized Riemann Hypothesis (GRH) for $L(f\otimes\chi,s)$, in the notation of Section 2 we have*

$$\operatorname*{Avg}_{\chi\in\Gamma_m}\Big(\overline{\mathfrak{g}(f\otimes\chi,s,z)}\,\mathfrak{g}(f\otimes\chi,s,z')\Big)-\sum_{(n,m)=1}\overline{\lambda_z(n)}\,\lambda_{z'}(n)n^{-2\sigma}\ll_{\varepsilon,R,T,f}m^{-\frac{\varepsilon}{2}}.\tag{5}$$

Moreover,

$$\lim_{m\to\infty}\operatorname*{Avg}_{\chi\in\Gamma_m}\Big(\overline{\mathfrak{g}(f\otimes\chi,s,z)}\,\mathfrak{g}(f\otimes\chi,s,z')\Big)=\sum_{n\geqslant 1}\overline{\mathfrak{l}_z(n)}\,\mathfrak{l}_{z'}(n)n^{-2\sigma}.$$

Proof. We notice that $\mathfrak{g}(f\otimes\chi,s,z)=\sum_{n\geqslant 1}\mathfrak{l}_z(n)\chi(n)n^{-s}$, where $\mathfrak{l}_z(n)$ are the coefficients of $\mathfrak{g}(f,s,z)$. We thus can deduce the theorem from [11, Theorem 1]. We can pass to the situation treated in [11] by omitting the summand corresponding to the trivial character χ_0 since in our case all the $\mathfrak{g}(f\otimes\chi,s,z)$ are holomorphic for $\operatorname{Re}s>\frac{1}{2}$. Thus, it is enough to prove that the family $\mathfrak{l}_{|z|\leqslant R}$ is uniformly admissible in the sense of Ihara and Matsumoto.

First of all, the property (A1), asserting that $\mathfrak{l}_{|z|\leqslant R}(n)\ll_\varepsilon n^\varepsilon$, follows from Lemma 2.1.

The property (A2) states that $\mathfrak{g}(f\otimes\chi,s,z)$ extend to holomorphic functions on $\operatorname{Re}s>\frac{1}{2}$ for any non trivial χ, which is true under GRH.

The property (A3) will be proven in the following lemma, which will be used again in Section 5.

Lemma 3.2. *Let f be a primitive form of weight N, and let χ be a primitive Dirichlet character of conductor m coprime with N. Then, assuming GRH for $L(f\otimes\chi,s)$, we have for $\operatorname{Re}s\geqslant\frac{1}{2}+\varepsilon$:*

$$\max(0,\log|\mathfrak{g}(f\otimes\chi,s,z)|)\ll_{\varepsilon,R}\ell(t)^{1-2\varepsilon}\ell(mNk)^{1-2\varepsilon},$$

where $\ell(x)=\log(|x|+2)$, $t=\operatorname{Im}s$.

Proof of the lemma. First, the following estimates hold [15, Theorems 5.17 and 5.19] for any s with $\frac{1}{2}<\operatorname{Re}s=\sigma\leqslant\frac{5}{4}$:

$$-\frac{L'(f\otimes\chi,s)}{L(f\otimes\chi,s)}=O\Big(\frac{1}{2\sigma-1}(\log\mathfrak{q}(f\otimes\chi,s))^{2-2\sigma}+\log\log\mathfrak{q}(f\otimes\chi,s)\Big),$$

and

$$\log L(f\otimes\chi,s)=O\Big(\frac{(\log\mathfrak{q}(f\otimes\chi,s))^{2-2\sigma}}{(2\sigma-1)\log\log\mathfrak{q}(f\otimes\chi,s)}+\log\log\mathfrak{q}(f\otimes\chi,s)\Big),$$

the implied constants being absolute.

Next, for the same range of s we have

$$\log\mathfrak{q}(f\otimes\chi,s)\ll\log(mNk)+\log(|t|+2)\ll\ell(mNk)+\ell(t).$$

Thus we see that

$$\log|\mathfrak{g}(f\otimes\chi,s,z)|=\log\Big|\exp\Big(\frac{iz}{2}\mathfrak{L}(f\otimes\chi,s)\Big)\Big|=\operatorname{Re}\Big(\frac{iz}{2}\mathfrak{L}(f\otimes\chi,s)\Big)\ll$$
$$\ll_R|\mathfrak{L}(f\otimes\chi,s)|,$$

so

$$\max(0, \log|\mathfrak{g}(f\otimes\chi, s, z)|) \ll_{\varepsilon,R} \ell(t)^{1-2\varepsilon}\ell(Nmk)^{1-2\varepsilon}.$$

If $\sigma \geqslant \frac{5}{4}$ a much simpler estimate suffices. Indeed, using the fact that [15, (5.25)]

$$-\frac{L'(f,s)}{L(f,s)} = \sum_n \frac{\Lambda_f(n)}{n^s} \quad \text{and} \quad \log L(f,s) = -\sum_n \frac{\Lambda_f(n)}{n^s \log n},$$

with $\Lambda_f(n)$ supported on prime powers and

$$\Lambda_f(p^n) = (\alpha_f(p)^n + \beta_f(p)^n)\log p,$$

we see that both $\frac{L'(f\otimes\chi,s)}{L(f\otimes\chi,s)}$ and $\log L(f\otimes\chi,s)$ are bounded by an absolute constant. Thus the conclusion of the lemma still holds in this case. $\square$

Thus Ihara and Matsumoto's property (A3) is established (with a stronger bound than required), since in our case N and k are fixed. So, the family we consider is indeed uniformly admissible. $\square$

Remark 3.3. We think that the estimate (5) should still be true if we omit the condition on m to be prime. To prove it one establishes an analogue of Lemma 3.2, replacing χ with the primitive character by which it is induced and estimating the bad factors of the L-function (with some additional work required when m is not coprime with N). Then one uses once again [11, Theorem 1], in which the first inequality is true without any restriction on the conductor.

Remark 3.4. The theorem should hold unconditionally for $\sigma = \operatorname{Re} s > 1$ by orthogonality of characters, all the series being absolutely convergent in this domain.

As a direct consequence, we obtain the following result on averages of the values of $\mathfrak{g}$. Put

$$\widetilde{M}_s(z_1,z_2) = \sum_{n=1}^{\infty} \mathfrak{l}_{z_1}(n)\mathfrak{l}_{z_2}(n)n^{-2s}.$$

Because of Lemma 2.1, the series converges uniformly and absolutely on $\operatorname{Re} s \geqslant \frac{1}{2}+\varepsilon$, $|z_1|, |z_2| \leqslant R$, defining a holomorphic function of s, z_1, z_2 for $\operatorname{Re} s > \frac{1}{2}$. Put

$$\psi_{z_1,z_2}(w) = \exp\Big(\frac{i}{2}(z_1\overline{w} + z_2 w)\Big).$$

Corollary 3.5. *Let m run over prime numbers. Then, assuming GRH,*

$$\lim_{m\to\infty} \operatorname*{Avg}_{\chi\in\Gamma_m} \psi_{z_1,z_2}(\mathfrak{L}(f\otimes\chi,s)) = \widetilde{M}_\sigma(z_1,z_2).$$

Proof. By definition, we have:

$$\psi_{z_1,z_2}(\mathfrak{L}(f\otimes\chi,s)) = \exp\Big(\frac{i}{2}z_1\overline{\mathfrak{L}(f\otimes\chi,s)}\Big)\exp\Big(\frac{i}{2}z_2\mathfrak{L}(f\otimes\chi,s)\Big) = \\ = \overline{\mathfrak{g}(f\otimes\chi,s,-\bar{z}_1)}\,\mathfrak{g}(f\otimes\chi,s,z_2).$$

By Theorem 3.1 we get

$$\lim_{m\to\infty} \operatorname*{Avg}_{\chi\in\Gamma_m} \psi_{z_1,z_2}(\mathfrak{L}(f\otimes\chi,s)) = \sum_{n\geqslant 1}\overline{\mathfrak{l}_{-\bar{z}_1}(n)}\,\mathfrak{l}_{z_2}(n)n^{-2\sigma}.$$

Lemma 2.2 implies that $\overline{\mathfrak{l}_{-\bar{z}}(n)} = \mathfrak{l}_z(n)$, so the corollary is proven. ☐

4. The Distribution of L-Values for Twists

Our next result concerns the distribution of the values of logarithmic derivatives and logarithms of L-functions of twists of a fixed modular form f. In this section the dependence on f in $\ll$ will be omitted.

Recall that we have defined

$$\widetilde{M}_s(z_1,z_2) = \sum_{n=1}^{\infty}\mathfrak{l}_{z_1}(n)\mathfrak{l}_{z_2}(n)n^{-2s},$$

the corresponding series being absolutely and uniformly convergent on $\operatorname{Re} s \geqslant \frac{1}{2}+\varepsilon$, $|z_1|\leqslant R$, $|z_2|\leqslant R$. For $\sigma\in\mathbb{R}$, we put $\widetilde{M}_\sigma(z) = \widetilde{M}_\sigma(z,\bar{z})$.

Define the family of additive characters

$$\psi_{z_1,z_2}(w) = \exp\Big(\frac{i}{2}(z_1\bar{w}+z_2 w)\Big).$$

We also let $\psi_z(w) = \psi_{z,\bar{z}}(w) = \exp(i\operatorname{Re}(z\bar{w}))$. Recall that the Fourier transform of $\phi:\mathbb{C}\to\mathbb{C}$, $\phi\in L^1$ is defined as

$$\mathscr{F}\phi(z) = \int_{\mathbb{C}}\phi(w)\psi_z(w)|dw| = \frac{1}{2\pi}\int_{\mathbb{C}}\phi(w)e^{i\operatorname{Re}(z\bar{w})}|dw| = \\ = \frac{1}{2\pi}\int_{\mathbb{R}^2}\phi(w)e^{i(xx'+yy')}\,dx\,dy,$$

where $|dw| = \frac{1}{2\pi}\,dx\,dy$, $x=\operatorname{Re} w$, $y=\operatorname{Im} w$, $x'=\operatorname{Re} z$, $y'=\operatorname{Im} z$.

The goal is to prove the following equidistribution result, which is an analogue of [11, Theorem 4].

Theorem 4.1. *Let* $\operatorname{Re} s = \sigma > \frac{1}{2}$ *and let* m *run over prime numbers. Let* Φ *be either a continuous function on* $\mathbb{C}$ *with at most exponential growth, that is,* $\Phi(w) \ll e^{a|w|}$ *for some* $a > 0$, *or the characteristic function of a bounded subset of* $\mathbb{C}$ *or of a complement of a bounded subset of* $\mathbb{C}$. *Define* M_σ *as the inverse Fourier transform of* $\widetilde{M}_\sigma(z)$, $M_\sigma(z) = \mathscr{F}\widetilde{M}_\sigma(-z)$. *Then under GRH for* $L(f \otimes \chi, s)$ *we have*

$$\lim_{m\to\infty} \operatorname*{Avg}_{\chi\in\Gamma_m} \Phi(\mathfrak{L}(f\otimes\chi, s)) = \int_{\mathbb{C}} M_\sigma(w)\Phi(w)|dw|. \tag{6}$$

Remark 4.2. We think that the above theorem should hold unconditionally for any $\sigma > 1$ and any continuous function Φ on $\mathbb{C}$, by virtue of Remarks 3.4 and (iv) of Corollary 4.12.

To prove this theorem we first construct the local M and $\widetilde{M}$-functions and establish their properties. We then obtain a convergence result for partial M-functions $M_{s,P}$ for finite sets of primes P to a global function M. This allows us to prove some crucial estimates for the growth of M. Finally, we deduce the global result using Corollary 3.5. Our approach is strongly influenced by that of Ihara and Matsumoto, the main ingredients being inspired by the results of Jessen and Wintner [18] that we have to adapt to our situation.

All the results below, except for the proof of Theorem 4.1 itself, do not depend on GRH.

4.1. The functions $M_{s,P}$ and $\widetilde{M}_{s,P}$

Let $\operatorname{Re} s = \sigma > 0$. Define the functions on $T_p = \mathbb{C}^1 = \{t \in \mathbb{C} : |t| = 1\}$ by

$$g_{s,p}(t) = \frac{-(\log p)\alpha(p)p^{-s}t}{1-\alpha(p)p^{-s}t} + \frac{-(\log p)\beta(p)p^{-s}t}{1-\beta(p)p^{-s}t},$$

and

$$G_{s,p}(t) = -\log(1-\alpha(p)p^{-s}t) - \log(1-\beta(p)p^{-s}t).$$

As before, we let $\mathfrak{g}_{s,p}$ denote either $g_{s,p}$ or $G_{s,p}$, depending on the case we consider. We note that the local factor of the L-function is 1 once $p^2 \mid N$, so we can omit such primes from our considerations.

Denote by $\mathfrak{f}_p(z)$ the expression

$$\frac{-(\log p)\alpha(p)z}{1-\alpha(p)z} + \frac{-(\log p)\beta(p)z}{1-\beta(p)z} \quad \text{or} \quad -\log(1-\alpha(p)z) - \log(1-\beta(p)z).$$

in the $\log'$ and $\log$ case respectively. Note that if $p \nmid N$,

$$\mathfrak{f}_p(z) = -\log p \cdot \frac{\eta_f(p)z - 2z^2}{1-\eta_f(p)z+z^2} \quad \text{or} \quad -\log(1-\eta_f(p)z+z^2)$$

respectively. The functions $\mathfrak{f}_p(z)$ are holomorphic in the open disc $|z|<1$. We obviously have $\mathfrak{g}_{s,p}(t)=\mathfrak{f}_p(p^{-s}t)$.

For a prime number p, let $T_p=\mathbb{C}^1$ be equipped with the normalized Haar measure $d^\times t=\dfrac{dt}{2\pi it}$. If P is a finite set of primes, we let $T_P=\prod\limits_{p\in P} T_p$ and we denote by $d^\times t_P$ the normalized Haar measure on T_P. Put also $\mathfrak{g}_{s,P}=\sum\limits_{p\in P}\mathfrak{g}_{s,p}$.

We introduce the local factors $\widetilde{M}_{s,p}(z_1,z_2)$ via

$$\widetilde{M}_{s,p}(z_1,z_2)=\sum_{r=0}^{+\infty}\mathfrak{l}_{z_1}(p^r)\mathfrak{l}_{z_2}(p^r)p^{-2rs}. \tag{7}$$

The series is absolutely and uniformly convergent on compacts in $\operatorname{Re}s>0$ by Lemma 2.1. Put $\widetilde{M}_{s,P}(z_1,z_2)=\prod\limits_{p\in P}\widetilde{M}_{s,p}(z_1,z_2)$. We also define $\widetilde{M}_{\sigma,p}(z)=$ $=\widetilde{M}_{\sigma,p}(z,\bar{z})$, and $\widetilde{M}_{\sigma,P}(z)=\widetilde{M}_{\sigma,P}(z,\bar{z})$.

Lemma 4.3. (i) *The function $\widetilde{M}_{s,P}(z_1,z_2)$ is entire in z_1,z_2.*

(ii) *We have*

$$\widetilde{M}_{s,p}(z_1,z_2)=\int_{\mathbb{C}^1}\exp\Bigl(\frac{i}{2}(z_1\mathfrak{g}_{s,p}(t^{-1})+z_2\mathfrak{g}_{s,p}(t))\Bigr)d^\times t.$$

In particular,

$$\widetilde{M}_{\sigma,p}(z_1,z_2)=\int_{\mathbb{C}^1}\psi_{z_1,z_2}(\mathfrak{g}_{\sigma,p}(t))d^\times t$$

and

$$\widetilde{M}_{\sigma,p}(z)=\int_{\mathbb{C}^1}\exp(i\operatorname{Re}(\mathfrak{g}_{\sigma,p}(t)\bar{z}))d^\times t.$$

(iii) *The "trivial" bound $|\widetilde{M}_{\sigma,p}(z)|\leqslant 1$ holds.*

Proof. (i) This is a direct corollary of the absolute and uniform convergence of the series of analytic functions (7), defining $\widetilde{M}_{s,p}(z_1,z_2)$.

(ii) It is clear from the definitions that

$$\exp\Bigl(\frac{iz}{2}\mathfrak{g}_{s,p}(t)\Bigr)=\sum_{r=0}^{\infty}\mathfrak{l}_z(p^r)(p^{-s}t)^r.$$

So, the statement is implied by the fact that $\widetilde{M}_{s,p}$ is the constant term of the Fourier series expansion of $\exp\Bigl(\frac{i}{2}(z_1\mathfrak{g}_{s,p}(t^{-1})+z_2\mathfrak{g}_{s,p}(t))\Bigr)$.

(iii) Obviously follows from (ii). $\square$

For the sake of convenience in what follows we will identify a function on $\mathbb{R}^2$ with the Radon measure or the tempered distribution it defines, when the latter make sense. We will also regard the Fourier transform or the convolution products as being defined via the corresponding distributions. We refer to [18, § 2, § 3] for more details.

Proposition 4.4. (i) *There exists a unique positive measure $M_{\sigma,P}$ of compact support and mass* 1 *on $\mathbb{C}\simeq\mathbb{R}^2$ such that*

$$M_{\sigma,P}(\Phi) = \int_{T_P} \Phi(\mathfrak{g}_{s,P}(t_P))d^\times t_P$$

for any continuous function Φ on $\mathbb{C}$.

(ii) $\mathscr{F}M_{\sigma,P} = \widetilde{M}_{\sigma,P}(z)$.

(iii) *There exists a set of primes $\mathscr{P}_f$ of positive density such that, for all $p\in\mathscr{P}_f$, $\widetilde{M}_{\sigma,p}(z) \ll_{p,\sigma} (1+|z|)^{-\frac{1}{2}}$.*

(iv) *Let P be a set of primes. If $|P\cap\mathscr{P}_f| > 4$, then $M_{\sigma,P}$ admits a continuous density (still denoted by $M_{\sigma,P}$) which is an L^1 function. The function $M_{\sigma,P}$ satisfies $M_{\sigma,P}(z) = M_{\sigma,P}(\bar{z}) \geqslant 0$.*

(v) *$M_{\sigma,P}$ is of class $\mathscr{C}^r$ once $|P\cap\mathscr{P}_f| > 2(r+2)$.*

Proof. (i) The uniqueness statement is obvious and the existence is given by the direct image measure $(\mathfrak{g}_{s,P})_*(d^\times t_P)$. The volume of an open set U of $\mathbb{R}^2$ is thus given by $M_{\sigma,P}(U) = \mathrm{Vol}(\mathfrak{g}_{s,P}^{-1}(U))$, therefore $M_{\sigma,P}$ has compact support equal to the image of $\mathfrak{g}_{s,P}$ and mass 1. From the formula $M_{s,P}(\Phi) = \int_{T_P} \Phi(\mathfrak{g}_{s,P}(t_P))d^\times t_P$, it is clear that $M_{s,P}$ depends only on σ, since Haar measures on T_P are invariant under multiplication by $p^{i\,\mathrm{Im}(s)}$.

(ii) From the definition of the convolution product we note that, regarded as distributions with compact support, $M_{\sigma,P} = *_{p\in P} M_{\sigma,p}$.

Next, $\mathscr{F}M_{\sigma,P} = \mathscr{F}(*_{p\in P}M_{\sigma,P}) = \prod_{p\in P}\mathscr{F}M_{\sigma,p}$. From Lemma 4.3 we see that $\widetilde{M}_{\sigma,P}(z_1,z_2) = M_{\sigma,P}(\psi_{z_1,z_2})$, and for the Fourier transforms of tempered distributions on $\mathbb{C}\simeq\mathbb{R}^2$ we have

$$\begin{aligned}
&\mathscr{F}M_{\sigma,p}(\phi) = \\
&= M_{\sigma,p}\Big(\int_{\mathbb{C}} \psi_z(w)\phi(w)|dw|\Big) = \int_{T_p}\int_{\mathbb{C}} \psi_{g_{s,p}(t)}(w)\phi(w)|dw|d^\times t = \\
&= \int_{\mathbb{C}}\int_{T_p} \psi_{g_{s,p}(t)}(w)\phi(w)d^\times t|dw| = \int_{\mathbb{C}} M_{\sigma,p}(\psi_z(w))\phi(w)|dw| = \\
&= \int_{\mathbb{C}} M_{\sigma,p}(\psi_w(z))\phi(w)|dw| = \int_{\mathbb{C}} \widetilde{M}_{\sigma,p}(w)\phi(w)|dw|.
\end{aligned}$$

We deduce that $\mathscr{F}M_{\sigma,P} = \widetilde{M}_{\sigma,P}(z)$.

(iii) This is the most delicate part. Unfortunately, we cannot apply Jessen—Wintner theorem [18, Theorem 13] to $\mathfrak{f}_p(z)$, since ρ_0 (in the notation of the latter theorem) depends on p. Therefore, we need to establish the following explicit version of their result.

Lemma 4.5. *Let $\rho > 0$ and let $F(z) = \sum_{k\geqslant 1} a_k z^k$ be absolutely convergent for $|z| < \rho + \varepsilon$, $\varepsilon > 0$. Let $S \subset \mathbb{C}$ denote the parametric curve $\{S(\theta)\}_{\theta\in[0,1]} = \{F(re^{2\pi i\theta})\}_{\theta\in[0,1]}$. Let D_r be the distribution on $\mathbb{C} = \mathbb{R}^2$ defined as the direct image of the normalized Haar measure on the circle of radius r in $\mathbb{C}$ by F and let $\widetilde{D}_r = \mathscr{F} D_r$ be its Fourier transform. Assume that $|a_1| \neq 0$. Then, if*

$$\rho''' = \frac{|a_1|}{\sqrt{2}\Big(\sum_{k\geqslant 2} k^3 |a_k| \rho^{k-2}\Big)},$$

for any $r < \rho_0 = \min(\rho, \rho''')$ we have $\widetilde{D}_r(z) \ll_{r,F} (1 + |z|)^{-\frac{1}{2}}$.

Proof. Our goal is to make the proof of [18, Theorem 13] explicit in order to be able to estimate ρ_0. To do so, we will verify the conditions of [18, Theorem 12] by proceeding in several steps.

First of all, we want to ensure that $F'(z) \neq 0$, and the curve S is Jordan. Put

$$\rho' = \frac{|a_1|}{\sqrt{2}\sum_{k\geqslant 2} k |a_k| \rho^{k-2}}.$$

If $r < \min(\rho, \rho')$, we have $F'(z) \neq 0$ for all $z \in \mathscr{D}_r = \overline{B(0,r)}$, and F is injective on $\mathscr{D}_r$. Indeed, either $|\operatorname{Re} a_1|$ or $|\operatorname{Im} a_1|$ is greater than $\dfrac{|a_1|}{\sqrt{2}}$. Without loss of generality we can suppose that $|\operatorname{Re} a_1| \geqslant \dfrac{|a_1|}{\sqrt{2}}$. Then

$$|\operatorname{Re} F'(z)| \geqslant |\operatorname{Re} a_1| - |z| \sum_{k\geqslant 2} k |a_k| \rho^{k-2} \geqslant \frac{|a_1|}{\sqrt{2}} - |z| \sum_{k\geqslant 2} k |a_k| \rho^{k-2} > 0$$

on $\mathscr{D}_r$, in particular $F'(z) \neq 0$. The sign of $\operatorname{Re} F'(z)$ does not change as the function is continuous, so once more, without loss of generality, we may assume that $\operatorname{Re} F'(z) > 0$. Then, for $z_1 \neq z_2$ two points in $\mathscr{D}_r$, we have by convexity of $\mathscr{D}_r$,

$$\operatorname{Re} \frac{F(z_2) - F(z_1)}{z_2 - z_1} = \int_0^1 \operatorname{Re} F'(z_1 + t(z_2 - z_1)) dt > 0,$$

which proves the injectivity. Thus F is a conformal transformation and S is a Jordan curve.

The next step is to get a condition for the curve S to be convex. We use a well-known criterion [29, Part 3, Chapter 3, 108], stating that S is convex if

$$\operatorname{Re}\frac{zF''(z)}{F'(z)} > -1$$

on $|z|=r$. The estimate

$$\left|\operatorname{Re}\frac{zF''(z)}{F'(z)}\right| \leqslant \frac{|zF''(z)|}{|F'(z)|} \leqslant \frac{|z|\sum_{k\geqslant 2} k(k-1)|a_k|\rho^{k-2}}{|a_1|-|z|\sum_{k\geqslant 2} k|a_k|\rho^{k-2}} \leqslant \frac{|z|\sum_{k\geqslant 2} k(k-1)|a_k|\rho^{k-2}}{|a_1|\left(1-\frac{1}{\sqrt{2}}\right)}$$

for $r<\min(\rho,\rho')$ implies that the condition is satisfied once the left-hand side is less than one, that is,

$$r < \rho'' = \frac{|a_1|(2-\sqrt{2})}{2\sum_{k\geqslant 2} k(k-1)|a_k|\rho^{k-2}}.$$

Now, the condition (i) of [18, Theorem 12] is satisfied for all $r<\rho$. As for (ii) we consider the function

$$g_\tau(\theta) = \sum_{k\geqslant 1} |a_k| r^k \cos 2\pi(k\theta+\gamma_k-\tau),$$

where $\tau\in[0,1)$ is fixed and $a_k=|a_k|e^{2\pi i\gamma_k}$. We have to prove that for r explicitly small enough, its second derivative has exactly two roots on $[0,1)$. We compute

$$\begin{aligned} h_\tau(\theta) &= -\frac{g_\tau''(\theta)}{4\pi^2 r} = \\ &= |a_1|\cos 2\pi(\theta+\gamma_1-\tau) + r\sum_{k\geqslant 2} j^2|a_k| r^{k-2}\cos 2\pi(k\theta+\gamma_k-\tau), \end{aligned}$$

so

$$\begin{aligned} h_\tau'(\theta) = -2\pi|a_1|\sin 2\pi(\theta+\gamma_1-\tau) - {}& \\ & - 2\pi r\sum_{k\geqslant 2} k^3|a_k| r^{k-2}\sin 2\pi(k\theta+\gamma_k-\tau). \end{aligned}$$

Take now

$$r < \frac{|a_1|}{\sqrt{2}\left(\sum_{k\geqslant 2} k^3|a_k|\rho^{k-2}\right)} = \rho'''.$$

Since $\sum_{k\geqslant 2} k^3|a_k|\rho^{k-2} \geqslant \sum_{k\geqslant 2} k^2|a_k|\rho^{k-2}$, the function h_τ can possibly have zeroes only on the two intervals (modulo 1) containing $\pm\frac{1}{4}-\gamma_1+\tau \bmod 1$ defined by the condition $|\cos 2\pi(\theta+\gamma_1-\tau)| < \frac{1}{\sqrt{2}}$. The same argument shows that h_τ is positive at $\theta=-\gamma_1+\tau \bmod 1$ and negative at $\theta=\frac{1}{2}+\tau-\gamma_1 \bmod 1$, and therefore it has at least one zero in each of these intervals.

On the other hand, when $|\cos 2\pi(\theta+\gamma_1-\tau)|<\frac{1}{\sqrt{2}}$, we see that

$$|h_\tau'(\theta)| \geqslant 2\pi|a_1|\cdot|\sin 2\pi(\theta+\gamma_1-\tau)| - 2\pi r\sum_{k\geqslant 2} k^3|a_k|r^{k-2} >$$
$$> 2\pi|a_1|\left(\sqrt{1-\tfrac{1}{2}}-\tfrac{1}{\sqrt{2}}\right)=0,$$

showing that there is exactly one zero of h_τ in each of the above intervals.

We thus can apply [18, Theorem 12], obtaining that the conclusion of the theorem holds for $r<\rho_0=\min(\rho,\rho',\rho'',\rho''')=\min(\rho,\rho''')$. $\square$

By [28, Corollary 2 of Theorem 4], there exists a set P of positive density such that, for all $p\in P$, $|\eta_f(p)|>1$. We apply the above lemma to the functions $F=\mathfrak{f}_p$, $p\in P$, defined by absolutely convergent series for $|z|<\rho+\varepsilon$, with $\rho=\varepsilon=\frac{1}{2}$, and to the radii $r_p=p^{-\sigma}$. In the log case, the coefficient $|a_1|$ of the lemma is $|\eta_f(p)|$, whereas we have for any i, $|a_i|\leqslant 2$. In the log$'$ case, the coefficients are all multiplied by $\log p$: $|a_1|$ is $|\eta_f(p)|\log p$ and $|a_i|\leqslant 2\log p$. Thus, for p such that $p\in P$ and

$$p^{-\sigma} < \frac{1}{8\sqrt{2}\sum_{k\geqslant 2} k^3 2^{-k}} = \frac{1}{204\sqrt{2}},$$

we have that $\widetilde{M}_{\sigma,p}(z)=O\left((1+|z|)^{-\frac{1}{2}}\right)$, proving thus (iii).

(iv), (v) By the Fourier inversion formula, we get $\mathscr{F}\widetilde{M}_{\sigma,P}(-z)=M_{\sigma,P}$. It is well-known [18, § 3] that $f=\mathscr{F}g$ is absolutely continuous and admits continuous density, once the integral $\int_{\mathbb{C}}|g(w)||dw|$ converges. Moreover, it possesses continuous partial derivatives of order $\leqslant p$, if the convergence holds for $\int_{\mathbb{C}}|z|^p|g(w)||dw|$. Thus, to deduce the regularity properties of $M_{\sigma,P}$ it suffices to bound the growth of $\widetilde{M}_{\sigma,P}(z)$.

For the primes $p \notin \mathscr{P}_f$, we use the trivial bound $|\widetilde{M}_{\sigma,p}(z)| \leqslant 1$ from Lemma 4.3. For all the other p the bound from (iii) can be applied.

Now, the identity $M_{\sigma,P}(z) = M_{\sigma,P}(\bar{z})$ is the consequence of (ii) together with the symmetry $\widetilde{M}_\sigma(z, \bar{z}) = \widetilde{M}_\sigma(\bar{z}, z)$. The positivity of $M_{\sigma,P}(z)$ follows from the definition $M_{\sigma,P}(U) = \mathrm{Vol}(\mathfrak{g}_{\sigma,P}^{-1}(U))$ together with the continuity that we have established. □

Let $\mathscr{P}$ denote the set of all prime numbers, $\mathscr{P}_x = \{p \in \mathscr{P} : p \leqslant x\}$.

Corollary 4.6. *Given $r > 0$, $y > 0$, there exists $C = C(y, r, f)$ such that $\widetilde{M}_{\sigma,\mathscr{P}_x \setminus \mathscr{P}_y}(z) = O((1+|z|)^{-r})$ and the function $M_{\sigma,\mathscr{P}_x \setminus \mathscr{P}_y}(z)$ is of class $\mathscr{C}^r$ for all $x \geqslant C$.*

Proof. This comes directly from the fact that $\mathscr{P}_f$ has positive density, implying that there exists C such that if $x \geqslant C$, then $(\mathscr{P}_x \setminus \mathscr{P}_y) \cap \mathscr{P}_f$ contains more than $2r + 4$ primes. □

Remark 4.7. The previous proposition is motivated by the following equidistribution result that is essentially implied by [5, Lemma 4.3.1] applied to $\Psi = \Phi \circ \mathfrak{g}_{\sigma,P}$:

$$\lim_{m\to\infty} \operatorname*{Avg}_{\chi\in\Gamma_m} \Phi(\mathfrak{L}_P(f \otimes \chi, s)) = \int_{T_P} \Phi(\mathfrak{g}_{\sigma,P}(t_P))d^\times t_P,$$

where Φ is a an arbitrary continuous function on $\mathbb{C}$, $\mathfrak{L}_P$ is either the logarithm or the logarithmic derivative of the corresponding partial product $\prod_{p\in P} L_p(f \otimes \chi, s)$ for $L(f \otimes \chi, s)$, and χ runs through all Dirichlet character of prime conductor $m \notin P$. Note a difference in the type of average considered in the aforementioned lemma with the one we use. The proof stays the same, being an application of Weyl's equidistribution criterion together with the orthogonality of characters.

Note, however, that it is not at all obvious to pass from the local equidistribution result to the global one. This also seems to give (after very significant effort) only a certain weaker form of global averaging results (e.g., [5], [9]). Following later papers by Ihara and Matsumoto, we use instead the convergence for particular test functions (quasi-characters, cf. Theorem 3.1) and then deduce the general case, using the information on the resulting distributions together with some general statements on convergence of measures.

4.2. Global results for $\widetilde{M}_\sigma$

Let us establish some global properties of $\widetilde{M}$, in particular the convergence of $\widetilde{M}_{\sigma,P}$ to $\widetilde{M}_\sigma$. From now on we assume that $\mathrm{Re}\, s = \sigma > \frac{1}{2}$, without mentioning it in each statement. Recall that $\widetilde{M}_\sigma(z)$ is defined as $\widetilde{M}_\sigma(z,\overline{z})$.

Proposition 4.8. (i) *The function* $\widetilde{M}_s(z_1,z_2)$ *is entire in* z_1, z_2.

(ii) *We have the Euler product expansion*

$$\widetilde{M}_s(z_1,z_2)=\prod_p \widetilde{M}_{s,p}(z_1,z_2),$$

which converges absolutely and uniformly on $\operatorname{Re} s \geqslant \frac{1}{2}+\varepsilon$ *and* $|z_1|, |z_2| \leqslant R$, *for any* $\varepsilon, R>0$.

(iii) $\widetilde{M}_\sigma(z)=O((1+|z|)^{-N})$ *for all* $N>0$.

Proof. (i) This is a direct corollary of the absolute and uniform convergence of the series of analytic functions, defining $\widetilde{M}_s(z_1,z_2)$.

(ii) To prove the uniform convergence of the infinite product it is enough to establish it for the sum $\sum\limits_p |\widetilde{M}_{s,p}(z_1,z_2)-1|$. By Lemma 2.1 we see that

$$|\widetilde{M}_{s,p}(z_1,z_2)-1| \leqslant \sum_{r=1}^{\infty} |\lambda_{z_1}(p^r)||\lambda_{z_2}(p^r)|p^{-2r\sigma} \ll_{\varepsilon',R} \sum_{r=1}^{\infty} p^{(2\varepsilon'-2\sigma)r} \leqslant$$

$$\leqslant \sum_{r=1}^{\infty} p^{(-1-\varepsilon)r} < 2p^{-1-\varepsilon},$$

which implies the convergence.

The limit of the infinite product equals $\widetilde{M}_s$. Indeed, the series for $\widetilde{M}_s$ converges absolutely and uniformly, thus the difference between $\widetilde{M}_s$ and the partial product over primes $p\leqslant x$, which is $\sum\limits_{n\in S_x} \lambda_{z_1}(n)\lambda_{z_2}(n)n^{-2s}$, where S_x is the set of integers n divisible by at least one prime strictly greater than x, tends to 0 as $x\to\infty$.

(ii) Note that for any two sets $P\subset P'$ of primes, any $z\in\mathbb{C}$,

$$|\widetilde{M}_{\sigma,P'}(z)| \leqslant |\widetilde{M}_{\sigma,P}(z)|.$$

Corollary 4.6 implies that one can find a finite set of primes P such that $\widetilde{M}_{\sigma,P}(z) \ll (1+|z|)^{-N}$. This is enough to conclude. $\square$

Remark 4.9. Along the same lines as in [10, 3.20], one proves a more precise estimate: $|\widetilde{M}_{\sigma,p}(z)-1| \ll |z|^2 p^{-2\sigma}$ in the log case, and

$$|\widetilde{M}_{\sigma,p}(z)-1| \ll |z|^2 p^{-2\sigma}\log p$$

in the $\log'$ case with absolute constants in $\ll$.

Remark 4.10. One should be able to write an explicit power series expansion of $\widetilde{M}_s(z_1,z_2)$ similar to the one in [11, § 4, Theorem $\widetilde{M}$].

4.3. Global results for M_σ

Proposition 4.11. *The sequence* $(M_{\sigma,\mathscr{P}_x}(z))_{x\gg 1}$ *converges uniformly (as continuous functions) to* $M_\sigma(z) := \mathscr{F}\widetilde{M}_\sigma(-z)$*. Moreover, for a fixed* y*, the sequence of continuous functions* $(M_{\sigma,\mathscr{P}_x\setminus\mathscr{P}_y})_{x\gg 1}$ *converges uniformly to the continuous function* $M_\sigma^{(y)} = *_{\mathscr{P}\setminus\mathscr{P}_y} M_{\sigma,p} := \mathscr{F}\Big(\prod_{p\in\mathscr{P}\setminus\mathscr{P}_y} \widetilde{M}_{\sigma,p}(-z)\Big)$*, and we have* $M_\sigma(z) = M_{\sigma,\mathscr{P}_y} * M_\sigma^{(y)}$.

Proof. First of all, the notation $x \gg 1$ is used to make sure that all the elements of the sequence are continuous functions.

Fix $\varepsilon > 0$. One can find a closed disk $\mathscr{D}_r$ and x' large enough, so that for all $P' \supset \mathscr{P}_{x'}$,

$$\int_{\mathbb{C}\setminus\mathscr{D}_r} |\widetilde{M}_{\sigma,P'}(w)||dw| < \varepsilon.$$

The sequence $(\widetilde{M}_{\sigma,\mathscr{P}_x}(z))_{x\gg 1}$ converges uniformly to $\widetilde{M}_\sigma(z) = \widetilde{M}_\sigma(z)$ on $\mathscr{D}_r$ by Proposition 4.8, thus we can find x'' large enough to guarantee for $x > \max(x', x'')$,

$$\|\mathscr{F}\widetilde{M}_\sigma(z) - \mathscr{F}\widetilde{M}_{\sigma,\mathscr{P}_x}(z)\|_\infty < 2\varepsilon.$$

This proves that the sequence $(\mathscr{F}\widetilde{M}_{\sigma,\mathscr{P}_x}(z))_{x\gg 1} = (M_{\sigma,\mathscr{P}_x}(-z))_{x\gg 1}$ converges uniformly to $\mathscr{F}\widetilde{M}_\sigma(z) = M_\sigma(-z)$.

The same arguments apply if we remove $\mathscr{P}_y$ from the set of all primes. Moreover, taking the Fourier transform of

$$\widetilde{M}_\sigma = \widetilde{M}_{\sigma,\mathscr{P}_y} \times \prod_{p\notin\mathscr{P}_y} \widetilde{M}_{\sigma,p},$$

we see that $M_\sigma(z) = M_{\sigma,\mathscr{P}_y} * (*_{\mathscr{P}\setminus\mathscr{P}_y} M_{\sigma,p})$. □

Corollary 4.12. *We have*

1) $M_\sigma(z) = M_\sigma(\bar{z}) \geqslant 0$;

2) $\int_{\mathbb{C}} M_\sigma(z)|dz| = 1$;

3) $M_\sigma(z) \in \mathscr{C}^\infty$ *and the partial derivatives of* $M_{\sigma,\mathscr{P}_x}$ *converge uniformly to those of* M_σ;

4) *If* $\sigma > 1$*, the support of* M_σ *is compact.*

Proof. (i) This is obvious from the corresponding properties of $M_{\sigma,P}$.

(ii) Using the identity $M_\sigma(z) = \mathscr{F}\widetilde{M}_\sigma(-z)$, we see that

$$\int_{\mathbb{C}} M_\sigma(z)|dz| = \widetilde{M}_\sigma(0) = 1.$$

(iii) We note that, given p, there exists y_0 such that for $p > y_0$, $M_{\sigma,\mathscr{P}_y}$ has continuous partial derivatives up to order p. Now, letting $D^{(a,b)} = \dfrac{\partial^{a+b}}{\partial^a z\, \partial^b \bar{z}}$, we have $D^{(a,b)}(f * g) = (D^{(a,b)} f) * g$, if f admits the corresponding partial derivative. The statement now follows from the uniform convergence of $M^{(y)}_{\sigma,\mathscr{P}_x \setminus \mathscr{P}_y}$ to $M^{(y)}_{\sigma}$.

(iv) Indeed, by the uniform convergence of $M_{\sigma,P}$ to M_σ, and the fact that the support of $M_{\sigma,P}$ is equal to the image of $\mathfrak{g}_{s,P}$, it is enough to prove that the latter is bounded for $\sigma > 1$. This is true since the series $\sum_p p^{-\sigma}$ converges for $\sigma > 1$. □

We will now obtain the rapid decay of M_σ à la Jessen—Wintner by proving the following proposition, which is crucial for the proof of the main theorem of this section.

Proposition 4.13. *For any $\lambda > 0$, $M_\sigma(z) = O_{\sigma,\lambda}(e^{-\lambda|z|^2})$, as $|z| \to \infty$. The same is true for all its partial derivatives.*

Proof. We adapt the proof of Jessen—Wintner [18, Theorem 16] to our specific case. The proof is based on an argument of Paley and Zygmund.

Let $\sigma > \frac{1}{2}$ and $\lambda > 0$ be fixed. Let $p_1 < \ldots < p_i \ldots$ denote the sequence of all prime numbers. Write $P_j = \{p_1, \ldots, p_j\}$. We have

$$\mathfrak{f}_p(z) = \sum_{i \geqslant 1} a_{i,p} z^i,$$

on the disk $B(0,1)$. By writing

$$1 - \eta_f(p) z + z^2 = (1 - \alpha_f(p) z)(1 - \beta_f(p) z),$$

where $|\alpha_f(p)|$ and $|\beta_f(p)|$ are less than or equal to 1, we see that for all i, $|a_{i,p}| \leqslant 2 \log p$ in the $\log'$ case and $\leqslant 2$ in the log case respectively.

Put $r_p = p^{-\sigma}$. Then the series $\sum_p |a_{1,p}|^2 r_p^2$ converges, so that we can find q such that

$$d = 1 - 2\lambda \sum_{p > p_q} |a_{1,p}|^2 r_p^2 > 0.$$

For $n > q$ let us look at the partial sums

$$s_n(\theta_1, \ldots, \theta_n) = \sum_{j=1}^{n} \mathfrak{f}_{p_j}(r_{p_j} e^{i\theta_j}) \quad \text{and} \quad t_n(\theta_{q+1}, \ldots, \theta_n) = \sum_{j=q+1}^{n} a_{1,p_j} r_{p_j} e^{i\theta_j},$$

where $\theta_j \in [0, 2\pi]$. We can bound the difference by

$$
\begin{aligned}
&|s_n(\theta_1, \dots, \theta_n) - t_n(\theta_{q+1}, \dots, \theta_n)| \leqslant \\
&\qquad \leqslant \left|\sum_{j=1}^{q} \mathfrak{f}_{p_j}(r_{p_j} e^{i\theta_j})\right| + \sum_{j=q+1}^{n} \sum_{k=2}^{\infty} |a_{k,p_j}| r_{p_j}^k \leqslant \\
&\qquad \leqslant \left|\sum_{j=1}^{q} \mathfrak{f}_{p_j}(r_{p_j} e^{i\theta_j})\right| + 2 \sum_{j=q+1}^{n} r_{p_j}^2 (1 - r_{p_j})^{-1} \log p_j \leqslant \\
&\qquad \leqslant \left|\sum_{j=1}^{q} \mathfrak{f}_{p_j}(r_{p_j} e^{i\theta_j})\right| + 8 \sum_{j=q+1}^{+\infty} r_{p_j}^2 \log p_j \leqslant \\
&\qquad \leqslant \sum_{j=1}^{q} \sup_{\vartheta_j \in [0,2\pi]} |\mathfrak{f}_{p_j}(r_{p_j} e^{i\vartheta_j})| + 8 \sum_{j=q+1}^{+\infty} r_{p_j}^2 \log p_j \ll A(q)
\end{aligned}
$$

as $(1 - r_{p_j})^{-1} \leqslant \dfrac{\sqrt{2}}{\sqrt{2} - 1}$. Here A depends only on q and not on n.

By an inequality of Jessen [17, p. 290—291], writing

$$|s_n|^2 \leqslant 2|s_n - t_n|^2 + 2|t_n|^2,$$

we obtain

$$
\begin{aligned}
&\int_{T_{P_n}} \exp(\lambda |s_n(\theta_1, \dots, \theta_n)|^2) d\theta_1 \dots d\theta_n \leqslant \\
&\qquad \leqslant e^{2\lambda A(q)^2} \int_{T_{P_{n,q}}} \exp(2\lambda |t_n(\theta_{q+1}, \dots, \theta_n)|^2) d\theta_{q+1} \dots d\theta_n \leqslant \\
&\qquad \leqslant \frac{e^{2\lambda A(q)^2}}{1 - 2\lambda \sum_{j=q+1}^{n} |a_{1,p}|^2 r_{p_j}^2} \leqslant e^{2\lambda A(q)^2} d^{-1} = K, \qquad (8)
\end{aligned}
$$

where $P_{n,q} = P_n \setminus P_q$. Noting that $M_{\sigma,P_n}(e^{\lambda|w|^2})$ is just the left-hand side of (8), we deduce:

$$M_{\sigma,P_n}(e^{\lambda|w|^2}) \leqslant K,$$

where K is independent of n. Thus by Fatou lemma and Proposition 4.11 we conclude that

$$\int_{\mathbb{C}} M_\sigma(w) e^{\lambda|w|^2} dw \leqslant K.$$

Let us take y such that $M_{\sigma,\mathscr{P}_y}$ is a continuous function. It is clear that if we remove all the terms corresponding to $p \leqslant y$, and take $q > y$

large enough we obtain exactly the same bound for the function $M_\sigma^{(y)} = = *_{p\in\mathscr{P}\setminus\mathscr{P}_y} M_{\sigma,p}$:

$$\int_{\mathbb{C}} M_\sigma^{(y)}(w) e^{\lambda|w|^2} dw \leqslant K.$$

If $\mathscr{D}_\rho = \overline{B(0,\rho)}$, $B = \overline{B(z,\rho)}$ denote the corresponding closed discs, $z \notin \mathscr{D}_\rho$, then

$$e^{\lambda(|z|-\rho)^2} \int_B M_\sigma^{(y)}(w)|dw| = \int_B e^{\lambda(|z|-\rho)^2} M_\sigma^{(y)}(w)|dw| \leqslant$$
$$\leqslant \int_B e^{\lambda|w|^2} M_\sigma^{(y)}(w)|dw| \leqslant K.$$

Let ρ be large enough, so that $\mathscr{D}_\rho$ contains the support of $M_{\sigma,\mathscr{P}_y}$. Then

$$\begin{aligned} M_\sigma(z) &= (M_{\sigma,\mathscr{P}_y} * M_\sigma^{(y)})(z) = \\ &= \int_{\mathbb{C}} M_{\sigma,\mathscr{P}_y}(w) M_\sigma^{(y)}(z-w)|dw| = \int_{\mathscr{D}_\rho} M_{\sigma,\mathscr{P}_y}(w) M_\sigma^{(y)}(z-w)|dw| \leqslant \\ &\leqslant \sup_{\mathscr{D}_\rho} M_{\sigma,\mathscr{P}_y}(w) \cdot \int_{\mathbb{C}} M_\sigma^{(y)}(z-w)|dw| \leqslant K e^{-\lambda(|z|-\rho)^2} \sup_{\mathscr{D}_\rho} M_{\sigma,\mathscr{P}_y}(w). \end{aligned}$$

As y, ρ, ρ are independent of z, we obtain that

$$M_\sigma(z) = O(e^{-\lambda|z|^2}).$$

According to Corollary 4.6, one can take y large enough so that $M_{\sigma,\mathscr{P}_y}$ has continuous partial derivatives of order up to p. We also have

$$D^{(a,b)}(f * g) = D^{(a,b)}(f) * g = f * D^{(a,b)}(g).$$

Thus, the same arguments as above imply that the required estimate holds for partial derivatives of $M_\sigma(z)$ of any order p. □

Corollary 4.14. *The functions $M_\sigma(z)$ and $\widetilde{M}_\sigma(z)$ belong to the Schwartz space, that is, they go to zero as $|z| \to \infty$ faster than any inverse power of $|z|$, as do all their derivatives.*

Proof. The statement is clear for $M_\sigma(z)$ by the above theorem. Now, $\widetilde{M}_\sigma(z) = \mathscr{F} M_\sigma(-z)$. Since $\mathscr{F}$ maps Schwartz functions to Schwartz functions the result follows. □

Corollary 4.15.

$$\widetilde{M}_\sigma(z_1, z_2) = \int_{\mathbb{C}} M_\sigma(w) \psi_{z_1,z_2}(w)|dw|.$$

Proof. Each side of the above equality is an entire function of z_1, z_2 (the left one by Proposition 4.8, the right one by Proposition 4.13). These functions are equal when $z_2 = \bar{z}_1$ by Proposition 4.11, thus they must coincide for any $z_1, z_2 \in \mathbb{C}$. □

Remark 4.16. The last corollary also follows from Theorem 4.1, however we prefer to give a direct proof.

4.4. Proof of Theorem 4.1

We will apply Lemma A from [11, § 5], which is a general result that allows to deduce from the convergence of averages for a special class of functions Φ, the same fact for more general Φ.

First of all, Corollaries 4.12 and 4.14 imply that M_σ is a good density function on $\mathbb{R}^2$ in the sense of Ihara and Matsumoto, that is, it is non-negative, real valued, continuous, with integral over $\mathbb{R}^2$ equal to 1, and such that both the function and its Fourier transform belong to $L^1 \cap L^\infty$.

By 3.5 the identity (6) holds for any additive character ψ_z of $\mathbb{C}$. Lemma A implies then that (6) is true for any bounded continuous Φ, for the characteristic function of any compact subset of $\mathbb{R}^2$ or of the complement of such a subset.

Now, take $\phi_0(r) = \exp(ar)$. Proposition 4.13 implies that

$$\int_{\mathbb{C}} M_\sigma(z)\phi_0(|z|)|dz|$$

converges. The same reasoning as in [11, § 5.3, Sublemma] allows us to see that $\mathrm{Avg}_{\chi \in \Gamma_m} \exp(a|\mathfrak{L}(f \otimes \chi, s)|) \ll 1$. This concludes the proof of Theorem 4.1.

5. Average on Primitive Forms

While working with modular forms it is analytically more natural to consider harmonic averages instead of usual ones. One introduces the harmonic weight

$$\omega(f) = \frac{\Gamma(k-1)}{(4\pi)^{k-1}(f,f)_N},$$

where

$$(f,f)_N = \int_{\Gamma_0(N)\backslash\mathscr{H}} |f(z)|^2 y^k \frac{dx\,dy}{y^2}$$

is the Petersson scalar product, $\mathscr{H}=\{z\in\mathbb{C}\colon\ \operatorname{Im} z>0\}$. We denote by $\operatorname*{Avg}^h_{f\in B_k(N)} G(f)$ the harmonic average

$$\operatorname*{Avg}^h_{f\in B_k(N)} G(f)=\sum_{f\in B_k(N)}\omega(f)G(f).$$

It can be proven [16, Corollary 2.10 for $m=n=1$] that for squarefree N we have

$$\sum_{f\in B_k(N)}\omega(f)=\frac{\varphi(N)}{N}+O\Big(\frac{\tau(N)^2\log(2N)}{Nk^{5/6}}\Big), \tag{9}$$

thus $\operatorname*{Avg}^h_{f\in B_k(N)}$ is an average operator when $\dfrac{\varphi(N)}{N}\to 1$.

One has the following interpretation of $\omega(f)$ via the symmetric square L-functions [16, Lemma 2.5]:

$$\omega(f)=\frac{2\pi^2}{(k-1)NL(\operatorname{Sym}^2 f,1)}. \tag{10}$$

Theorem 5.1. *Assume that N is a prime number and that k is fixed. Let $0<\varepsilon<\frac{1}{2}$ and $T,R>0$. Let $s=\sigma+it$ belong to the domain $\sigma\geqslant\varepsilon+\frac{1}{2}$, $|t|\leqslant T$, and z and z' to a disc $\mathscr{D}_R$. Then, assuming GRH for $L(f,s)$, for any $\delta>0$ we have*

$$\operatorname*{Avg}^h_{f\in B_k(N)}\big(\overline{\mathfrak{g}(f,s,z)}\,\mathfrak{g}(f,s,z')\big)-\\ -\sum_{n,m\in\mathbb{N}} n^{-\bar{s}}m^{-s}\sum_{\substack{x\in J(n)\cap J(m)\\(nm,N)=1}}\overline{c_{z,x}(n)}\,c_{z',x}(m)\ll_{\varepsilon,R,T,\delta,k} N^{-\varepsilon/2+\delta},$$

and

$$\lim_{N\to+\infty}\operatorname*{Avg}^h_{f\in B_k(N)}\big(\overline{\mathfrak{g}(f,s,z)}\,\mathfrak{g}(f,s,z')\big)=\\ =\sum_{n,m\in\mathbb{N}} n^{-\bar{s}}m^{-s}\sum_{x\in J(n)\cap J(m)}\overline{c_{z,x}(n)}c_{z',x}(m).$$

The convergence of the series is on the right-hand sides is uniform and absolute in the above domains without the assumption of GRH.

Remark 5.2. In contrast to the situation, considered in Theorem 3.1, we see that the average depends both on $\operatorname{Re} s$ and $\operatorname{Im} s$. In fact, the independence of $\operatorname{Im} s$ in the case of averages with respect to characters is the corollary of the invariance of Haar measures on $\mathbb{C}^1$ under rotations.

Corollary 5.3. *Under the conditions of the previous theorem we have*

$$\lim_{N\to+\infty} \operatorname*{Avg}^h_{f\in B_k(N)} \psi_{z_1,z_2}(\mathfrak{L}(f,s)) = \widetilde{M}^h_s(z_1,z_2) =$$
$$= \sum_{n,m\in\mathbb{N}} n^{-\bar{s}} m^{-s} \sum_{x\in J(n)\cap J(m)} c_{z_1,x}(n) c_{z_2,x}(m).$$

Proof. We have $\psi_{z_1,z_2}(\mathfrak{L}(f,s)) = \overline{\mathfrak{g}(f,s,-\overline{z}_1)}\, \mathfrak{g}(f,s,z_2)$, so

$$\lim_{N\to+\infty} \operatorname*{Avg}^h_{f\in B_k(N)} \psi_{z_1,z_2}(\mathfrak{L}(f,s)) = \sum_{n,m\in\mathbb{N}} n^{-\bar{s}} m^{-s} \sum_{x\in J(n)\cap J(m)} \overline{c_{-\bar{z}_1,x}(n)} c_{z_2,x}(m).$$

The corollary follows from the equality $\overline{c_{-\bar{z},x}(n)} = c_{z,x}(n)$, which is implied by Lemma 2.2. □

5.1. Naive approach

In this subsection we try to estimate the average in a naive way via Euler products. This approach works for $\operatorname{Re} s$ large enough and gives a formula which turns out to be valid for more general s. The intermediate calculations will be used again in Section 5.3. All the estimates are written assuming only that N is squarefree and not assuming that k is fixed until the very end of Section 5.3.

We have

$$\operatorname*{Avg}^h_{f\in B_k(N)} (\overline{\mathfrak{g}(f,s,z)}\, \mathfrak{g}(f,s,z')) = \sum_{f\in B_k(N)} \omega(f) \sum_{n,m\geqslant 1} n^{-\bar{s}} m^{-s} \overline{\mathfrak{l}_z(n)}\, \mathfrak{l}_{z'}(m).$$

Let $\tau_k(n) = |\{(d_1,\dots,d_k)\in\mathbb{N}^k : d_1 \dots m d_k = n\}|$. We will use a version of the Petersson formula proven in [16, Corollary 2.10]. Note that our weights are slightly different from those used in [16], we follow instead [31] in our normalization.

Proposition 5.4. *If N is squarefree, $(m,N)=1$, $(n,N^2)\,|\,N$, then*

$$S(m,n) = \sum_{f\in B_k(N)} \omega(f)\eta_f(m)\eta_f(n) = \frac{\varphi(N)}{N}\delta(m,n) + \Delta(m,n),$$

where $\delta(m,n)$ is the Kronecker symbol and

$$\Delta(m,n) = O\big(k^{-\frac{5}{6}} (mn)^{\frac{1}{4}} N^{-1} (n,N)^{-1/2} \tau(N)^2 \tau_3((m,n)) \log(2mnN)\big),$$

the implied constant being absolute.

The conditions of this proposition are in particular satisfied once $(nm,N)=1$. We will also use the following trivial bound, when $(m,N)\neq 1$:

$$|S(m,n)|\leqslant \sum_{f\in B_k(N)}\omega(f)\frac{\tau(m)\tau(n)}{\sqrt{(m,N)}}=$$
$$=\Big(\frac{\varphi(N)}{N}+O\Big(\frac{\tau(N)^2\log(2N)}{Nk^{5/6}}\Big)\Big)\frac{\tau(m)\tau(n)}{\sqrt{(m,N)}},\qquad (11)$$

which holds by virtue of (9) and the fact that $|\eta_f(m)|\leqslant\dfrac{\tau(m)}{\sqrt{(m,N)}}$ since N is squarefree. Obviously, the corresponding bound is also true if we assume $(n,N)\neq 1$ instead of $(m,N)\neq 1$.

Remark 5.5. In what follows, one can possibly soften our restrictions on N (in particular, remove the assumption that $N\to\infty$) by using more elaborate bounds on the sums in the case when $(mn,N)\neq 1$, applying directly the construction of an explicit basis of $S_k(N)$ from $B_k(N)$, in a way similar to [16, Proposition 2.6].

Using the above estimates, we can write

$$\operatorname*{Avg}^h_{f\in B_k(N)}(\overline{\mathfrak{g}(f,s,z)}\mathfrak{g}(f,s,z'))=\sum_{f\in B_k(N)}\omega(f)\sum_{n,m}n^{-\bar{s}}m^{-s}\overline{\mathfrak{l}_z(n)}\mathfrak{l}_{z'}(m)=$$
$$=\sum_{n,m}n^{-\bar{s}}m^{-s}\sum_{f\in B_k(N)}\omega(f)\overline{\mathfrak{l}_z(n)}\mathfrak{l}_{z'}(m)=$$
$$=\sum_{n,m}n^{-\bar{s}}m^{-s}\sum_{x\in J_N(n),y\in J_N(m)}\overline{c^N_{z,x}(n)}c^N_{z',y}(m)\sum_{f\in B_k(N)}\omega(f)\eta_f(x)\eta_f(y)+$$
$$=\sum_{n,m}n^{-\bar{s}}m^{-s}\sum_{\substack{x\in J_N(n),y\in J_N(m)\\(xy,N)=1}}\overline{c^N_{z,x}(n)}c^N_{z',y}(m)\Big(\delta(x,y)\frac{\varphi(N)}{N}+\Delta(x,y)\Big)=$$
$$+\sum_{n,m}n^{-\bar{s}}m^{-s}\sum_{\substack{x\in J_N(n),y\in J_N(m)\\(xy,N)\neq 1}}\overline{c^N_{z,x}(n)}c^N_{z',y}(m)\sum_{f\in B_k(N)}\omega(f)\eta_f(x)\eta_f(y)=$$
$$=\frac{\varphi(N)}{N}\sum_{n,m}n^{-\bar{s}}m^{-s}\sum_{\substack{x\in J_N(n)\cap J_N(m)\\(x,N)=1}}\overline{c^N_{z,x}(n)}c^N_{z',x}(m)+$$
$$+\sum_{n,m}n^{-\bar{s}}m^{-s}\sum_{\substack{x\in J_N(n),y\in J_N(m)\\(xy,N)=1}}\overline{c^N_{z,x}(n)}c^N_{z',y}(m)\Delta(x,y)+$$
$$+\sum_{n,m}n^{-\bar{s}}m^{-s}\sum_{\substack{x\in J_N(n),y\in J_N(m)\\(xy,N)\neq 1}}\overline{c^N_{z,x}(n)}c^N_{z',y}(m)S(x,y).$$

The fact that the sum can be subdivided into three parts will be justified by the absolute convergence of the series for $\operatorname{Re} s$ large enough.

Put

$$\widetilde{M}(s) = \sum_{n,m} n^{-\bar{s}} m^{-s} \sum_{\substack{x \in J_N(n) \cap J_N(m) \\ (x,N)=1}} \overline{c^N_{z,x}(n)}\, c^N_{z',x}(m).$$

Let us first note that the sum does not depend on N, since $c^N_{z,x}(n) = c_{z,x}(n)$ if $(n,N) = 1$, and the coefficient $c^N_{z,x}(n)$ vanishes, once we have both $(x,N) = 1$, and $(n,N) \neq 1$. This allow us to write

$$\widetilde{M}(s) = \sum_{n,m} n^{-\bar{s}} m^{-s} \sum_{\substack{x \in J(n) \cap J(m) \\ (nm,N)=1}} \overline{c_{z,x}(n)}\, c_{z',x}(m).$$

Our goal is to verify that $\widetilde{M}(s)$ gives the principal term of the asymptotic behaviour of $\operatorname{Avg}^h_{f \in B_k(N)} (\overline{\mathfrak{g}(f,s,z)}\, \mathfrak{g}(f,s,z'))$. If $m \notin I(n)$, which is equivalent to $I(m) \neq I(n)$, the term $\overline{c_{z,x}(n)}\, c_{z',x}(m)$ vanishes. Therefore,

$$\widetilde{M}(s) = \sum_{\substack{n \in \mathbb{N},\ (nm,N)=1 \\ m \in I(n)}} n^{-\bar{s}} m^{-s} \sum_{x \in J(n) \cap J(m)} \overline{c_{z,x}(n)}\, c_{z',x}(m).$$

Let us define $r_-(n)$ to be the largest integer whose square divides n, and $r_+(n)$ to be the least positive integer whose square is divisible by n. So, if $n = p_1^{k_1} \ldots p_l^{k_l}$, we have

$$n = p_1^{k_1 \bmod 2} \ldots p_l^{k_l \bmod 2} r_-(n)^2, \quad r_+(n)^2 = p_1^{k_1 \bmod 2} \ldots p_l^{k_l \bmod 2} n,$$

and the squarefree part of n is equal to

$$p_1^{k_1 \bmod 2} \ldots p_l^{k_l \bmod 2} = \frac{r_+(n)}{r_-(n)}.$$

Using this notation, we can write for $s = \sigma + it$

$$\widetilde{M}(s) = \sum_{n \geqslant 1} \sum_{r \geqslant 1} n^{-\sigma+it} \left(\frac{r_+(n)}{r_-(n)}\right)^{-\sigma-it} r^{-2s} \sum_{\substack{x \in J(n) \cap J(m) \\ (mn,N)=1}} c_{z,x}(n) c_{z',x}\left(\frac{r_+(n) r^2}{r_-(n)}\right) =$$

$$= \sum_{n,r \geqslant 1} r_+(n)^{-2\sigma} r_-(n)^{2it} r^{-2s} \sum_{\substack{x \in J(n) \cap J(m) \\ (mn,N)=1}} c_{z,x}(n) c_{z',x}\left(\frac{r_+(n) r^2}{r_-(n)}\right), \quad (12)$$

where $m = \dfrac{r_+(n) r^2}{r_-(n)}$, so

$$|\widetilde{M}(s)| \leqslant \sum_{n,r \geqslant 1} (r_+(n))^{-2\sigma} r^{-2\sigma} \sum_{\substack{x \in J(n) \\ (nr,N)=1}} |c_{z,x}(n)| \left| c_{z',x}\left(\frac{r_+(n) r^2}{r_-(n)}\right) \right|.$$

There are $2^{\omega(n)} - 1 = 2^l - 1$ different n giving the same $r_+(n)$. As $\omega(n) \ll \dfrac{\log n}{2+\log\log n}$ by (4), so $2^l \ll_\varepsilon n^\varepsilon$, using Lemma 2.1 and (3) we see that the sum $\widetilde{M}(s)$ converges absolutely for $\operatorname{Re} s > 1/2$:

$$|\widetilde{M}(s)| \ll_\varepsilon \sum_{n,r\geqslant 1} n^\varepsilon \cdot n^{-2\sigma} \cdot r^{-2\sigma} \cdot n^\varepsilon \cdot n^\varepsilon \cdot r^{2\varepsilon} \cdot n^\varepsilon = \sum_{n\geqslant 1} n^{-2\sigma+4\varepsilon} \sum_{r\geqslant 1} r^{-2\sigma+2\varepsilon}.$$

Let us now see what happens with the error term. If we put

$$\Delta(s) = \sum_{n,m\geqslant 1} n^{-\bar{s}} m^{-s} \sum_{\substack{x\in J_N(n),y\in J_N(m)\\ (xy,N)=1}} \overline{c^N_{z,x}(n)}\, c^N_{z',y}(m)\Delta(x,y),$$

from the Proposition 5.4 together with the estimate $\tau_3(n) \leqslant \tau(n)^3 \ll_\varepsilon n^\varepsilon$, and Lemma 2.1 we conclude that

$$|\Delta(s)| \ll_\varepsilon \frac{\tau(N)^2 \log N}{Nk^{5/6}} \sum_{m,n\geqslant 1} (mn)^{-\sigma+\frac{1}{4}+\varepsilon}.$$

In a similar way, putting

$$\Delta'(s) = \sum_{n,m\geqslant 1} n^{-\bar{s}} m^{-s} \sum_{\substack{x\in J_N(n),y\in J_N(m)\\ (xy,N)\neq 1}} \overline{c^N_{z,x}(n)}\, c^N_{z',y}(m)S(x,y),$$

we get

$$|\Delta'(s)| \ll_\varepsilon \frac{1}{\sqrt{p_{\min}(N)}}\Big(\frac{\varphi(N)}{N} + O\Big(\frac{\tau(N)^2\log(2N)}{Nk^{5/6}}\Big)\Big) \sum_{m,n\geqslant 1} (mn)^{-\sigma+\varepsilon},$$

where $p_{\min}(N)$ is the least prime factor of N.

These bounds only make sense for $\sigma = \operatorname{Re} s > 5/4$, when the series converge. For these values of s we conclude that the error terms tend to 0, once $p_{\min}(N) \to \infty$ (recall that we assume N to be squarefree). In the next section we are going to show how the estimates can be pushed to the left of $\operatorname{Re} s > 5/4$.

5.2. Integral representation

We introduce the following notation. Let $0 < \varepsilon' < \varepsilon < \dfrac{1}{2}$, $s \in \mathbb{C}$ with $\sigma = \operatorname{Re} s \geqslant \dfrac{1}{2} + \varepsilon$, $c > \max(0, 1-\sigma)$, $X \geqslant 1$ a parameter to be specified later. The symbol $\ll$ will depend on ε, R, and T but this dependence will not be explicitly indicated. As before, we assume only that N is squarefree (and not necessarily prime), and we do not suppose k to be fixed. We will write $\mathfrak{g}$ to denote $\mathfrak{g}(f, s, z)$ when no ambiguity is possible.

We use the techniques from [11], though it would be possible to employ the approximate functional equations instead, since they are available in our case. First, we establish the analogues of the propositions proven in [11, § 2.2].

Lemma 5.6. (i) *For* $\operatorname{Re} s \geqslant \frac{1}{2} + \varepsilon$ *we have* $\mathfrak{g} = \mathfrak{g}_+ - \mathfrak{g}_-$*, where the holomorphic functions* $\mathfrak{g}_+$ *and* $\mathfrak{g}_-$ *are defined by*

$$\mathfrak{g}_+(f, s, z, X) = \frac{1}{2\pi i} \int\limits_{\operatorname{Re} w = c} \Gamma(w) \mathfrak{g}(f, s + w, z) X^w \, dw,$$

and

$$\mathfrak{g}_-(f, s, z, X) = \frac{1}{2\pi i} \int\limits_{\operatorname{Re} w = \varepsilon' - \varepsilon} \Gamma(w) \mathfrak{g}(f, s + w, z) X^w \, dw.$$

(ii) *The function* $\mathfrak{g}_+$ *has a Dirichlet series expansion*

$$\mathfrak{g}_+ = \sum_{n=1}^{\infty} \mathfrak{l}_z(n) e^{-\frac{n}{X}} n^{-s}$$

which is absolutely and uniformly convergent on compacts in $\mathbb{C}$*.*

Proof. The first statement admits exactly the same proof as the corresponding part of [11, Proposition 2.2.1] with Ihara and Matsumoto's property (A3) being replaced by Lemma 3.2 in our case.

As for the second statement, we have the Dirichlet series expansion

$$\mathfrak{g}(f, s, z) = \sum_{n=1}^{\infty} \mathfrak{l}_z(n) n^{-s}.$$

Taking into account that $\sigma + c > 1$, we see that

$$\mathfrak{g}(f, s + w, z) = \sum_{n=1}^{\infty} \mathfrak{l}_z(n) n^{-s-w}$$

is absolutely and uniformly convergent with respect to $\operatorname{Im} w$ on $\operatorname{Re} w = c$. Exchanging the integration and summation and using

$$\frac{1}{2\pi i} \int\limits_{\operatorname{Re} w = c} \Gamma(w) a^{-w} \, dw = e^{-a},$$

we obtain the desired expansion. The absolute and uniform convergence is clear for Lemma 2.1. □

In what follows we will estimate $\mathfrak{g}_+$ on average, which will give the main term, the function $\mathfrak{g}_-$ will on the contrary be estimated individually for each f. The following lemma bounds $\mathfrak{g}_-$ in terms of the parameter X.

Lemma 5.7. *Let* $\operatorname{Re} s \geqslant 1/2 + \varepsilon$. *Then for any* $f \in B_k(N)$, $0 < \varepsilon' < \varepsilon$, $T > 0$, *for* $|\operatorname{Im}(s)| \leqslant T$ *we have*

$$|\mathfrak{g}_-(f, s, z, X)| \ll_{\varepsilon'} (NkX)^{\varepsilon'} X^{-\varepsilon}.$$

Proof. Once again our proof largely mimics that of [11, Proposition 2.2.13]. We need to estimate the integral

$$\mathfrak{g}_-(f, s, z, X) = \frac{1}{2\pi i} \int\limits_{\operatorname{Re} w = \varepsilon' - \varepsilon} \Gamma(w) \mathfrak{g}(f, s + w, z) X^w \, dw.$$

Clearly, $|X^w| = X^{\varepsilon' - \varepsilon}$ and it is well-known [11, (2.2.9)] that

$$\Gamma(w) \ll |\operatorname{Im} w|^{c - 1/2} \exp\Big(-\frac{\pi}{2} |\operatorname{Im}(w)|\Big),$$

when $|\operatorname{Im} w| \geqslant 1$, $\operatorname{Re} w \leqslant c$, so in our case $\Gamma(w) \ll \exp(-|\operatorname{Im}(w)|)$. Lemma 3.2 ensures that, putting $u = \operatorname{Im}(w)$ and $t = \operatorname{Im}(s)$, we have

$$\log |\mathfrak{g}(f, s + w, z)| \ll \ell(kN)^{1 - 2\varepsilon'} \ell(t + u)^{1 - 2\varepsilon'}.$$

Therefore, there exists $C = C(T, \varepsilon')$ such that

$$\begin{aligned} |\mathfrak{g}(f, s + w)| &\leqslant \exp\Big(C \ell(Nk)^{1 - 2\varepsilon'} (\log(|u| + 1))^{1 - 2\varepsilon'}\Big) \leqslant \\ &\leqslant \exp\Big(C \ell(Nk)^{1 - 2\varepsilon'} \log(|u| + 1)\Big). \end{aligned}$$

So, by comparison with the Γ-integral, we have

$$\begin{aligned} |\mathfrak{g}_-(f, s + w, z, X)| &\ll X^{\varepsilon' - \varepsilon} \int\limits_0^{+\infty} e^{-u} (u + 1)^{C \ell(Nk)^{1 - 2\varepsilon'}} du \ll \\ &\ll X^{\varepsilon' - \varepsilon} \Gamma(C \ell(Nk)^{1 - 2\varepsilon'} + 1) \ll \\ &\ll X^{\varepsilon' - \varepsilon} \exp(C \ell(Nk)^{1 - 2\varepsilon'} \log(C \ell(Nk)^{1 - 2\varepsilon'})) \ll \\ &\ll X^{\varepsilon' - \varepsilon} \exp(C' \ell(Nk)^{1 - 2\varepsilon'} \log(\ell(Nk))) \ll \\ &\ll_{\varepsilon'} X^{\varepsilon' - \varepsilon} \exp(\varepsilon' \ell(Nk)) \ll X^{\varepsilon' - \varepsilon} (Nk)^{\varepsilon'}, \end{aligned}$$

since for Nk large enough depending on T and ε',

$$C' \ell(Nk)^{-2\varepsilon'} \log(\ell(Nk)) < \varepsilon'$$

holds. □

5.3. Averaging

We now go back to averaging over primitive forms. We denote for simplicity $\mathfrak{g}=\mathfrak{g}(s,f,z)$, $\mathfrak{g}'=\mathfrak{g}(s,f,z')$ and we adopt similar notation for $\mathfrak{g}_\pm$ and $\mathfrak{g}'_\pm$.

First of all, using the decomposition established in Section 5.2, we note that

$$\operatorname*{Avg^h}_{f\in B_k(N)}(\overline{\mathfrak{g}}\mathfrak{g}') =$$
$$= \operatorname*{Avg^h}_{f\in B_k(N)}(\overline{\mathfrak{g}}_+\mathfrak{g}'_+) - \operatorname*{Avg^h}_{f\in B_k(N)}(\overline{\mathfrak{g}}_+\mathfrak{g}'_-) - \operatorname*{Avg^h}_{f\in B_k(N)}(\overline{\mathfrak{g}}_-\mathfrak{g}'_+) + \operatorname*{Avg^h}_{f\in B_k(N)}(\overline{\mathfrak{g}}_-\mathfrak{g}'_-).$$

Our first goal is to prove that the average

$$\operatorname*{Avg^h}_{f\in B_k(N)}(\overline{\mathfrak{g}}_+\mathfrak{g}'_+) = \sum_{f\in B_k(N)}\omega(f)\sum_{n,m\geqslant 1} n^{-\bar{s}}m^{-s}\overline{\mathfrak{l}_z(n)}\mathfrak{l}_{z'}(m)e^{-\frac{n+m}{X}}.$$

gives the main term of the asymptotic behaviour. The calculations of Section 5.1 allow us to decompose the above average as follows:

$$\operatorname*{Avg^h}_{f\in B_k(N)}(\overline{\mathfrak{g}}_+\mathfrak{g}'_+) = \widetilde{M}(s,X)+\Delta(s,X)+\Delta'(s,X), \tag{13}$$

with

$$\widetilde{M}(s,X) = \frac{\varphi(N)}{N}\sum_{n,m} n^{-\bar{s}}m^{-s}e^{-\frac{n+m}{X}}\sum_{\substack{x\in J_N(n)\cap J_N(m)\\(x,N)=1}}\overline{c^N_{z,x}(n)}\,c^N_{z',x}(m),$$

$$\Delta(s,X) = \sum_{n,m} n^{-\bar{s}}m^{-s}e^{-\frac{n+m}{X}}\sum_{\substack{x\in J_N(n),y\in J_N(m)\\(xy,N)=1}}\overline{c^N_{z,x}(n)}\,c^N_{z',y}(m)\Delta(x,y),$$

$$\Delta'(s,X) = \sum_{n,m} n^{-\bar{s}}m^{-s}e^{-\frac{n+m}{X}}\sum_{\substack{x\in J_N(n),y\in J_N(m)\\(xy,N)\neq 1}}\overline{c^N_{z,x}(n)}\,c^N_{z',y}(m)S(x,y).$$

Noting that $0<1-e^{-a}<\min(a,1)$ and fixing any $\alpha>0$, we see that

$$|\widetilde{M}(s)-\widetilde{M}(s,X)| =$$
$$= \frac{\varphi(N)}{N}\left|\sum_{n,m} n^{-\bar{s}}m^{-s}(1-e^{-\frac{n+m}{X}})\sum_{\substack{x\in J_N(n)\cap J_N(m)\\(x,N)=1}}\overline{c^N_{z,x}(n)}c^N_{z',x}(m)\right| \leqslant$$
$$\leqslant \frac{\varphi(N)}{N}\sum_{\substack{n\leqslant\alpha X\\m\leqslant\alpha X}} n^{-\sigma}m^{-\sigma}\frac{n+m}{X}\sum_{\substack{x\in J_N(n)\cap J_N(m)\\(x,N)=1}}|c^N_{z,x}(n)||c^N_{z',x}(m)| +$$
$$+\frac{\varphi(N)}{N}\sum_{\substack{n\geqslant\alpha X\\ \text{or}\\ m\geqslant\alpha X}} n^{-\sigma}m^{-\sigma}(1-e^{-\frac{n+m}{X}})\sum_{\substack{x\in J_N(n)\cap J_N(m)\\(x,N)=1}}|c^N_{z,x}(n)||c^N_{z',x}(m)|.$$

The calculations of Section 5.1 together with the observation that (in the notation of (12)) $r^2 r_+(b)^2 = mn$ result in the following bound valid for any $\varepsilon'' > 0$:

$$\sum_{\substack{n\geqslant \alpha X\\ \text{or}\\ m\geqslant \alpha X}} n^{-\sigma} m^{-\sigma}(1-e^{-\frac{n+m}{X}}) \sum_{\substack{x\in J_N(n)\cap J_N(m)\\ (x,N)=1}} |c^N_{z,x}(n)||c^N_{z',x}(m)| \ll_{\varepsilon''}$$

$$\ll_{\varepsilon''} \sum_{(rs)^2\geqslant \alpha X} (rs)^{-2\sigma+\varepsilon''} \ll_{\varepsilon''} \sum_{r\geqslant \sqrt{\alpha X}} r^{-2\sigma+\varepsilon''} \ll_{\varepsilon''} (\alpha X)^{1/2-\sigma+\varepsilon''/2},$$

while the absolute convergence of the series for $\widetilde{M}(s)$ implies

$$\sum_{\substack{n\leqslant \alpha X\\ m\leqslant \alpha X}} n^{-\sigma} m^{-\sigma} \frac{n+m}{X} \sum_{\substack{x\in J_N(n)\cap J_N(m)\\ (x,N)=1}} |c^N_{z,x}(n)||c^N_{z',x}(m)| \ll \alpha.$$

Taking ε'' small enough so that $\beta = 1/2-\sigma+\varepsilon''/2 < 0$ and α satisfying $\alpha = (\alpha X)^{\beta}$, we finally see that

$$|\widetilde{M}(s) - \widetilde{M}(s,X)| \ll_{\varepsilon''} \frac{\varphi(N)}{N} X^{\frac{1/2-\sigma+\varepsilon''/2}{1/2+\sigma-\varepsilon''/2}} \leqslant \frac{\varphi(N)}{N} X^{\varepsilon''/2-\varepsilon}. \tag{14}$$

Now, let us turn to the second and the third terms in (13). Once again, applying the estimates from Section 5.1 we see that for any $\varepsilon'' > 0$

$$|\Delta(s,X)| \ll_{\varepsilon''} \frac{\tau(N)^2 \log N}{Nk^{5/6}} \sum_{m,n\geqslant 1} (mn)^{-\sigma+\frac{1}{4}+\varepsilon''} e^{-\frac{m+n}{X}},$$

$$|\Delta'(s,X)| \ll_{\varepsilon''}$$

$$\ll_{\varepsilon''} \frac{1}{\sqrt{p_{\min}(N)}}\left(\frac{\varphi(N)}{N} + O\left(\frac{\tau(N)^2\log(2N)}{Nk^{5/6}}\right)\right) \sum_{m,n\geqslant 1} (mn)^{-\sigma+\varepsilon''} e^{-\frac{m+n}{X}}.$$

Bounding the sums via the corresponding improper integrals (cf. [11, proof of Proposition 2.2.13]), we get

$$|\Delta(s,X)| \ll_{\varepsilon''} \frac{\tau(N)^2 \log N}{Nk^{5/6}} X^{3/2+2\varepsilon''-2\varepsilon}, \tag{15}$$

$$|\Delta'(s,X)| \ll_{\varepsilon''} \frac{1}{\sqrt{p_{\min}(N)}}\Big(\frac{\varphi(N)}{N} + O\Big(\frac{\tau(N)^2\log(2N)}{Nk^{5/6}}\Big)\Big) X^{1+2\varepsilon''-2\varepsilon}. \tag{16}$$

In what follows, we will choose X (as a function of N) in such a way that the right-hand sides in (14), (15), and (16) tend to 0. With this choice of X, taking $z = z'$ and using the absolute convergence of $\widetilde{M}(s)$, we obtain

$$\operatorname*{Avg}^h_{f\in B_k(N)} |\mathfrak{g}_+|^2 \ll_{\varepsilon''} 1, \qquad \operatorname*{Avg}^h_{f\in B_k(N)} |\mathfrak{g}'_+|^2 \ll_{\varepsilon''} 1$$

Let us estimate the remaining terms involving $\mathfrak{g}_-$ and $\mathfrak{g}'_-$. By Lemma 5.7 and (9)

$$\operatorname*{Avg}^h_{f\in B_k(N)} |\mathfrak{g}_-|^2 \ll_{\varepsilon',T} (NkX)^{2\varepsilon'} X^{-2\varepsilon} \sum_{f\in B_k(N)} \omega(f) \leqslant$$
$$\leqslant \Big(\frac{\varphi(N)}{N} + O\Big(\frac{\tau(N)^2\log(2N)}{Nk^{5/6}}\Big)\Big)(NkX)^{2\varepsilon'} X^{-2\varepsilon}.$$

We apply the Cauchy—Schwartz to get

$$|\operatorname*{Avg}^h_{f\in B_k(N)} (\overline{\mathfrak{g}}_+\mathfrak{g}'_-)| + |\operatorname*{Avg}^h_{f\in B_k(N)} (\overline{\mathfrak{g}}_-\mathfrak{g}'_+)| + |\operatorname*{Avg}^h_{f\in B_k(N)} (\overline{\mathfrak{g}}_-\mathfrak{g}'_-)| \ll$$
$$\ll_{\varepsilon'} \Big(\frac{\varphi(N)}{N} + O\Big(\frac{\tau(N)^2\log(2N)}{Nk^{5/6}}\Big)\Big)(NkX)^{2\varepsilon'} X^{-\varepsilon}, \quad (17)$$

since $(NkX)^{2\varepsilon'} \geqslant (NkX)^{\varepsilon'}$ and $X^{-2\varepsilon} \leqslant X^{-\varepsilon}$.

Let us now turn to the case considered in the theorem, by assuming that k is fixed and $N = p$ is prime. Assuming that $\varepsilon'' < 2\varepsilon$, we have

$$|\widetilde{M}(s) - \widetilde{M}(s, X)| \ll_{\varepsilon''} X^{\varepsilon''/2-\varepsilon},$$
$$|\Delta(s, X)| \ll_{\varepsilon''} \frac{\log p}{p} X^{3/2+2\varepsilon''-2\varepsilon},$$
$$|\Delta'(s, X)| \ll_{\varepsilon''} \frac{1}{\sqrt{p}} X^{1+2\varepsilon''-2\varepsilon},$$
$$|\operatorname*{Avg}^h_{f\in B_k(N)} (\overline{\mathfrak{g}}_+\mathfrak{g}'_-)| + |\operatorname*{Avg}^h_{f\in B_k(N)} (\overline{\mathfrak{g}}_-\mathfrak{g}'_+)| + |\operatorname*{Avg}^h_{f\in B_k(N)} (\overline{\mathfrak{g}}_-\mathfrak{g}'_-)| \ll_{\varepsilon',k} p^{2\varepsilon'} X^{2\varepsilon'-\varepsilon}.$$

Taking $X = p^{1/2}$, we see that the above bounds lead to

$$\Big|\operatorname*{Avg}^h_{f\in B_k(N)} \overline{\mathfrak{g}(f,s,z)}\, \mathfrak{g}(f,s,z') - \widetilde{M}(s)\Big| \ll_\delta p^{-\varepsilon/2+\delta},$$

where δ, which depends on ε' and ε'', can be taken arbitrarily small.

The second part of the theorem follows from the first.

6. Open Questions and Remarks

This section is devoted to a series of questions and remarks to complement the results of the paper. We hope to address at least some of them in subsequent articles. We start by the topics discussed in Section 3.

Question 6.1. Can Theorem 3.1 be proven in a greater generality?

For example, one can consider L-functions of more general automorphic cusp forms and the average taken with respect to their twists by Hecke characters of imaginary quadratic number fields or algebraic function fields with a fixed place at infinity. As indicated in [11], going beyond imaginary quadratic number fields seems to be tricky since it involves essentially new problems related to the presence of non-trivial units. One can also consider averages over quadratic characters in the spirit of [27].

Question 6.2. What is a version of Theorem 4.1 without assuming GRH?

The unconditional results [10, Theorem 1] and [12, Theorem 1.1] suggest that it should be possible to prove similar statements in our case.

Question 6.3. Prove an analogue of Theorem 4.1 for modular forms in the other situations within the framework of the cases (A), (B), (C) discussed in the introduction.

Some results in this direction were established by Mastsumoto in [23] in the case (C), that is, the equidistribution of $L(f, \sigma + it)$, when σ is fixed and $t \in \mathbb{R}$ varies. It seems, however, that, even when considering averages of Dirichlet L-functions conditionally on GRH, this question has not been fully investigated, the most advanced results having been obtain only in the case (A).

Question 6.4. Carry out a more in-depth study of the functions M and $\widetilde{M}$.

In the case of Dirichlet characters this was done in [6], [7]. One should be able to write down an explicit power series expansion of $\widetilde{M}_s(z_1, z_2)$ in the variables z_1, z_2, establish its analytic continuation, study its growth, its zeroes, etc.

We next switch to the case of averages with respect to primitive forms of Section 5, where the results are far less complete.

Question 6.5. Can one obtain Theorem 5.1 with weaker assumptions on N? Can we let k tend to infinity, while N is fixed? Can we let $k + N \to \infty$?

By following carefully the proof of Theorem 5.1, one can see that the limit statement is still true when $N = 1$ and $k \to \infty$. Indeed, in this case Δ' is not present and the parameter X cas be chosen to be equal to $k^{1/2}$. This suggests that some greater generality should be possible. The idea would be to use better bounds on averages of the Fourier coefficients of cusp forms with indices not coprime with N, which should be possible by a careful treatement of an explicit basis of the space of old forms in the spirit of [16].

Question 6.6. Prove an unconditional version of Theorem 5.1.

Surprisingly enough, a crude reasoning with Euler products does not seem to work even for $\operatorname{Re} s > 1$. An unconditional version for $\operatorname{Re} s > 1/2$

will certainly be tricky to obtain even if one only considers characters ψ_z as in[10] and [12].

Question 6.7. Is it possible to establish value distribution results in the case of harmonic averages over the set of primitive forms?

The reason we could not carry out the study analogous to that of Section 4 is the absence of a local theory (at least in a straight-forward way). Indeed, the $\widetilde{M}_s$ do not seem to admit an Euler product in this case. One could hope to rely on the interpretation of $\omega(f)$ via the symmetric square L-functions (10), though there does not seem to be an easy way to do that.

Question 6.8. Can one remove the harmonic weights in Theorem 5.1?

At least two approaches are available. The papers [16], [21], [22] address a similar issue in different situations by using the interpretation (10) of the weights via $L(\mathrm{Sym}^2 f, 1)$.

A more conceptual way would be to construct the local theory first. The results of Serre [32] on the equidistribution of the eigenvalues of Hecke operators T_p suggest that the local picture should be fairly clear. This would allow to establish the value distribution results missing in the case of harmonic averages. We plan to address this question in a forthcoming paper.

Question 6.9. Can one prove Theorem 5.1 in greater generality for other types of automorphic forms?

The first obvious step would be establishing it for $L(f \otimes \chi, s)$. For more general L-functions an appropriate trace formula would be necessary to replace Petersson formula.

Question 6.10. What is a function field version of Theorem 5.1?

The GRH being known in this case, unconditional results should not be very difficult to establish along the lines of this paper, once proper definitions are given.

Question 6.11. Establish the properties of $\widetilde{M}$ functions in the case of averages with respect to primitive forms.

Some peculiarities do arise compared to the case of characters. For example, $\widetilde{M}_s(z_1, z_2)$ is no longer holomorphic in s, since the average does depend on s and $\bar{s}$. The function is still entire in z_1, z_2 for fixed s. Establishing its explicit power series expansion, analytic continuation, etc. seems to be of interest. The growth properties of $\widetilde{M}$ seem to be much more delicate in our case, since they are proven using local results in the situation of Ihara and Matsumoto.

Question 6.12. Write an adelic version of Ihara's and Matsumoto's results, as well as of our results in the setting of modular forms.

This might shed some light on and give a better understanding of the functions M, $\widetilde{M}$, as well as of the relation of the global theory to the local one. One might also hope to be able to deal with the problems related to units in the number field case (cf. Question 6.1).

Question 6.13. What are the arithmetic implications of our results?

The results of Ihara and Matsumoto give us a better understanding of the behaviour of the Euler—Kronecker constants of cyclotomic fields. More generally, since the log case of averaging results for $\mathbb{Q}$ concerns, in particular, zeta functions of cyclotomic fields $\zeta_{\mathbb{Q}(\zeta_m)}(s)$, which are simply the products of $L(s,\chi)$ over primitive Dirichlet characters of conductors dividing m, the results of Ihara and Matsumoto can be seen as a first step in the development of a finer version of the asymptotic theory of global fields from [34], that gives non-trivial results for abelian extensions. This is not the case in [34], since infinite global fields, containing infinite abelian subfields are asymptotically bad in the terminology of loc. cit.

When one takes averages with respect to primitive forms, the results are close in spirit to the asymptotic study of zeta functions (see [33] for their definition) of modular curves $X_0(N)$, which can be written as

$$\zeta_{X_0(N)}(s) = \prod_{f \in B_2(N)} L(f,s)$$

(this function is the normalized L-function of the Jacobian variety of $X_0(N)$). Establishing a precise relation boils down to answering Question 6.8.

Note that even a cruder version of the asymptotic theory in the spirit of [34] has not been developed in this case. In the function field case this was to a significant extent done in [35]. A higher dimensional asymptotic theory in the characteristic zero case is yet to be constructed.

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Dense families of modular curves, prime numbers and uniform symmetric tensor rank of multiplication in certain finite fields

(with S. Ballet)

Abstract. We obtain new uniform bounds for the symmetric tensor rank of multiplication in finite extensions of any finite field $\mathbb{F}_p$ or $\mathbb{F}_{p^2}$ where p denotes a prime number $\geqslant 5$. In this aim, we use the symmetric Chudnovsky-type generalized algorithm applied on sufficiently dense families of modular curves defined over $\mathbb{F}_{p^2}$ attaining the Drinfeld—Vladuts bound and on the descent of these families to the definition field $\mathbb{F}_p$. These families are obtained thanks to prime number density theorems of type Hoheisel, in particular a result due to Dudek (2016).

1. Introduction

1.1. Notation

Let $\mathbb{F}_q$ be a finite field with q elements where q is a prime power and let $\mathbb{F}_{q^n}$ be an $\mathbb{F}_q$-extension of degree n. The multiplication of two elements of $\mathbb{F}_{q^n}$ is an $\mathbb{F}_q$-bilinear map from $\mathbb{F}_{q^n} \times \mathbb{F}_{q^n}$ onto $\mathbb{F}_{q^n}$. It can be considered as an $\mathbb{F}_q$-linear map from the tensor product $\mathbb{F}_{q^n} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$ onto $\mathbb{F}_{q^n}$. Consequently it can be also viewed as an element T of $\mathbb{F}_{q^n}^\star \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}^\star \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$ where $\mathbb{F}_{q^n}^\star$ denotes the dual of $\mathbb{F}_{q^n}$. More precisely, when T is expressed as

$$T = \sum_{i=1}^{r} x_i^\star \otimes y_i^\star \otimes c_i, \tag{1}$$

where $x_i^\star \in \mathbb{F}_{q^n}^\star$, $y_i^\star \in \mathbb{F}_{q^n}^\star$ and $c_i \in \mathbb{F}_{q^n}$, the following holds for any $x, y \in \mathbb{F}_{q^n}$:

$$x \cdot y = T(x \otimes y) = \sum_{i=1}^{r} x_i^\star(x) y_i^\star(y) c_i.$$

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Definition 1. The minimal number of summands in a decomposition of the multiplication tensor T is called the tensor rank of the multiplication in the extension field $\mathbb{F}_{q^n}$ (or bilinear complexity of the multiplication) and is denoted by $\mu_q(n)$:

$$\mu_q(n) = \min\left\{r \mid T = \sum_{i=1}^{r} x_i^\star \otimes y_i^\star \otimes c_i\right\}.$$

It is known that the tensor T can have a symmetric decomposition:

$$T = \sum_{i=1}^{r} x_i^\star \otimes x_i^\star \otimes c_i. \tag{2}$$

Definition 2. The minimal number of summands in a symmetric decomposition of the multiplication tensor T is called the symmetric tensor rank of the multiplication (or the symmetric bilinear complexity of the multiplication) and is denoted by $\mu_q^{\mathrm{sym}}(n)$:

$$\mu_q^{\mathrm{sym}}(n) = \min\left\{r \mid T = \sum_{i=1}^{r} x_i^\star \otimes x_i^\star \otimes c_i\right\}.$$

From an asymptotical point of view, let us define the following

$$M_q^{\mathrm{sym}} = \limsup_{k\to\infty} \frac{\mu_q^{\mathrm{sym}}(k)}{k}, \tag{3}$$

$$m_q^{\mathrm{sym}} = \liminf_{k\to\infty} \frac{\mu_q^{\mathrm{sym}}(k)}{k}. \tag{4}$$

1.2. Known results

The original algorithm of D. V. and G. V. Chudnovsky introduced in [11] is symmetric by definition and leads to the following results from [3], [8] and [7]:

Theorem 1. *Let q be a prime power and let $n > 1$ be an integer. Let $F/\mathbb{F}_q$ be an algebraic function field of genus g and N_k be the number of places of degree k in $F/\mathbb{F}_q$. Suppose $F/\mathbb{F}_q$ is such that $2g+1 \leqslant q^{\frac{n-1}{2}}(q^{\frac{1}{2}}-1)$ then*:

i) *if $N_1 > 2n+2g-2$, then*

$$\mu_q^{\mathrm{sym}}(n) \leqslant 2n+g-1,$$

ii) *if $N_1 + 2N_2 > 2n+2g-2$ and there exists a non-special divisor of degree $g-1$, then*

$$\mu_q^{\mathrm{sym}}(n) \leqslant 3n+2g.$$

Theorem 2. *Let q be a power of a prime p and let n be an integer. Then the symmetric tensor rank $\mu_q^{\mathrm{sym}}(n)$ is linear with respect to the extension degree; more precisely, there exists a constant C_q such that for any integer $n>1$,*

$$\mu_q^{\mathrm{sym}}(n) \leqslant C_q n.$$

From different versions of symmetric algorithms of Chudnovsky type applied to good towers of algebraic function fields of type Garcia—Stichtenoth attaining the Drinfeld—Vladuts bounds of order one, two or four, different authors have obtained uniform bounds for the tensor rank of multiplication, namely general expressions for C_q, such as the following best currently published estimates:

Theorem 3. *Let $q=p^r$ be a power of a prime p and let n be an integer >1. Then*:

(i) *If $q=2$, then $\mu_q^{\mathrm{sym}}(n)\leqslant 15.46n$ (cf.* [6, Corollary 29] *and* [10]);

(ii) *If $q=3$, then $\mu_q^{\mathrm{sym}}(n)\leqslant 7.732n$ (cf.* [6, Corollary 29] *and* [10]);

(iii) *If $q\geqslant 4$, then* $\mu_q^{\mathrm{sym}}(n)\leqslant 3\left(1+\dfrac{\frac{4}{3}p}{q-3+2(p-1)\dfrac{q}{q+1}}\right)n$ (*cf.* [7]);

(iv) *If $p\geqslant 5$, then* $\mu_p^{\mathrm{sym}}(n)\leqslant 3\left(1+\dfrac{8}{3p-5}\right)n$ (*cf.* [7]);

(v) *If $q\geqslant 4$, then* $\mu_{q^2}^{\mathrm{sym}}(n)\leqslant 2\left(1+\dfrac{p}{q-3+(p-1)\dfrac{q}{q+1}}\right)n$ (*cf.* [1] *and* [7]);

(vi) *If $p\geqslant 5$, then* $\mu_{p^2}^{\mathrm{sym}}(n)\leqslant 2\left(1+\dfrac{2}{p-\dfrac{33}{16}}\right)n$ (*cf.* [7]).

1.3. New results

The main goal of the paper is to improve the upper bounds for $\mu_q^{\mathrm{sym}}(n)$ from the previous theorem for the assertions concerning the extensions of finite fields $\mathbb{F}_{p^2}$ and $\mathbb{F}_p$ where p is a prime number. One of main ideas used in this paper was introduced in [4] by the first author thanks to the use of the Chebyshev Theorem (or also called the Bertrand Postulat) to bound the gaps between prime numbers. More precisely, the aim was to construct families of modular curves $\{X_i\}$ with increasing genus g_i attaining the Drinfeld-Vladut bound as dense as possible. This means that these families of modular curves have the maximum possible ratio of the number of $\mathbb{F}_{p^2}$-rational points to the genus and such that the sequence of their genera is as dense as possible, namely $\lim\limits_{i\to\infty}\dfrac{g_{i+1}}{g_i}=1$.

Later, motivated by [4], the approach of using such bounds on gaps between prime numbers (e.g. Baker-Harman-Pintz) was also used in the preprint [13] in order to improve the upper bounds of $\mu_{p^2}^{\mathrm{sym}}(n)$ where p is a prime number. In our paper, we improve all the known uniform upper bounds for $\mu_{p^2}^{\mathrm{sym}}(n)$ and $\mu_p^{\mathrm{sym}}(n)$ for $p \geqslant 5$. This article is an expansion of a paper which was presented at The Tenth International Workshop on Coding and Cryptography (WCC17) [9].

2. New upper bounds

In this section, we give new better upper bounds for the symmetric tensor rank of multiplication in certain extensions of finite fields $\mathbb{F}_{p^2}$ and $\mathbb{F}_p$. In order to do that, we construct suitable families of modular curves defined over $\mathbb{F}_{p^2}$ and $\mathbb{F}_p$. In this aim, we need explicit prime number density theorems, usually called theorems of type Hoheisel. In particular, by a result of Baker, Harman and Pintz [2, Theorem 1] established in 2001 and by a recent result established by Dudek [12] in 2016, we directly deduce the following result:

Theorem 6. *Let l_k be the k-th prime number. Then there exist real numbers $\alpha < 1$ and x_α such that the difference between two consecutive prime numbers l_k and l_{k+1} satisfies*

$$l_{k+1} - l_k \leqslant l_k^\alpha$$

for any prime $l_k \geqslant x_\alpha$.

In particular, one can take $\alpha = \frac{21}{40}$ with the value of x_α that can in principle be determined effectively, or $\alpha = \frac{2}{3}$ with $x_\alpha = \exp(\exp(33.217))$.

Proof. It is known that there exists a real number x_α such that for all $x > x_\alpha$, the interval $[x - x^\alpha, x]$ with $\alpha = \frac{21}{40}$ contains prime numbers by a result of Baker, Harman and Pintz [2, Theorem 1]. In particular, if $l_k > x_\alpha$ denotes the k-th prime number, it means that the interval $[l_k, l_k + l_k^\alpha]$ contains the $k+1$-th prime number l_{k+1}. Moreover, the value of x_α can in principle be determined, according to the authors. However, to our knowledge, this computation has not been realized yet.

For a bigger $\alpha = \frac{2}{3}$, Dudek obtained recently in [12, Theorem 1.1] an explicit bound $x_\alpha \geqslant \exp(\exp(33.217))$. More precisely, Dudek proves that there exists a prime between cubes, namely the interval $[n^3, (n+1)^3]$ contains a prime number for sufficiently large numbers n. From this result, we can directly deduce that there exists a prime in the interval $[x, x+3x^{\frac{2}{3}}]$ for all sufficiently large x. Moreover, he makes the result explicit, in that he determines numerically a lower bound for which this result is valid, namely for $x \geqslant \exp(\exp(33.217))$. Then, if we put $[x, x+3x^{\frac{2}{3}}] = [x, x+x^{\alpha}]$, we deduce that $\alpha = \frac{2}{3} + \varepsilon$ with $\varepsilon <$ $< \frac{\ln 3}{\exp(\exp(33.217))}$ for any $x > \exp(\exp(33.217))$. □

2.1. The case of the quadratic extensions of prime fields

Proposition 7. *Let $p \geqslant 5$ be a prime number, and let x_α be the constant from Theorem 6.*

(i) *If $p \neq 11$, then for any integer $n \geqslant \frac{p-3}{2}x_\alpha + \frac{p+1}{2}$ we have*

$$\mu^{\mathrm{sym}}_{p^2}(n) \leqslant 2\Big(1+\frac{1+\varepsilon_p(n)}{p-3}\Big)n - \frac{(1+\varepsilon_p(n))(p+1)}{p-3} - 1,$$

where $\varepsilon_p(n) = \Big(\frac{2n}{p-3}\Big)^{\alpha-1}$.

(ii) *For $p = 11$ and $n \geqslant (p-3)x_\alpha + p - 1 = 8x_\alpha + 10$ we have*

$$\mu^{\mathrm{sym}}_{p^2}(n) \leqslant 2\Big(1+\frac{1+\varepsilon_p(n)}{p-3}\Big)n - \frac{2(1+\varepsilon_p(n))(p-1)}{p-3},$$

where $\varepsilon_p(n) = \Big(\frac{n}{p-3}\Big)^{\alpha-1}$.

(iii) *Asymptotically the following inequality holds for any $p \geqslant 5$:*

$$M^{\mathrm{sym}}_{p^2} \leqslant 2\Big(1+\frac{1}{p-3}\Big).$$

Proof. First, let us consider the characteristic p such that $p \neq 11$. Then it is known ([16, Corollary 4.1.21] and [15, proof of Theorem 3.9]) that the modular curve $X_k = X_0(11l_k)$, where l_k is the k-th prime number, is of genus $g_k = l_k$ and satisfies

$$N_1(X_k(\mathbb{F}_{p^2})) \geqslant (p-1)(g_k+1),$$

where $N_1(X_k(\mathbb{F}_{p^2}))$ denotes the number of rational points over $\mathbb{F}_{p^2}$ of the curve X_k. Let us consider an integer $n > 1$. Then there exist two consecutive prime numbers l_k and l_{k+1} such that

$$(p-1)(l_{k+1}+1) > 2n + 2l_{k+1} - 2 \qquad (5)$$

and

$$(p-1)(l_k+1) \leqslant 2n + 2l_k - 2 \qquad (6)$$

(here we use the fact that $p \geqslant 5$). Let us consider the algebraic function field $F_{k+1}/\mathbb{F}_{p^2}$ associated to the curve X_{k+1} of genus l_{k+1} defined over $\mathbb{F}_{p^2}$. Denoting by $N_i(F_k/\mathbb{F}_{p^2})$ the number of places of degree i of $F_k/\mathbb{F}_{p^2}$, we get

$$N_1(F_{k+1}/\mathbb{F}_{p^2}) \geqslant (p-1)(l_{k+1}+1) > 2n + 2l_{k+1} - 2.$$

We also know that $l_{k+1} - l_k \leqslant l_k^\alpha$, when $l_k \geqslant x_\alpha$ by Theorem 6. Thus $l_{k+1} \leqslant (1+\varepsilon(l_k))l_k$, with $\varepsilon(l_k) = l_k^{\alpha-1}$. It is easy to check that the inequality $2g+1 \leqslant q^{\frac{n-1}{2}}(q^{\frac{1}{2}}-1)$ of Theorem 1 holds for any prime power $q \geqslant 5$. Indeed, it is enough to verify that

$$q^{l_k\frac{p-3}{4}+\frac{p-1}{4}}(q^{\frac{1}{2}}-1) \geqslant 2(1+\varepsilon(l_k))l_k + 1,$$

which is true since

$$q^{x\frac{p-3}{4}+\frac{p-1}{4}}(q^{\frac{1}{2}}-1) - 4x - 1 \geqslant 0$$

for any $x \geqslant 0$.

Thus, for any integer $n \geqslant \dfrac{p-3}{2}x_\alpha + \dfrac{p+1}{2}$ the function field $F_{k+1}/\mathbb{F}_{p^2}$ satisfies Theorem 1, so

$$\mu_{p^2}^{\mathrm{sym}}(n) \leqslant 2n + l_{k+1} - 1 \leqslant 2n + (1+\varepsilon(l_k))l_k - 1,$$

with $l_k \leqslant \dfrac{2n}{p-3} - \dfrac{p+1}{p-3}$ by (6).

Let us remark that, as $l_k \leqslant \dfrac{2n}{p-3}$, $\varepsilon(l_k) \leqslant \varepsilon_p(n) = (\dfrac{2n}{p-3})^{\alpha-1}$, which gives the first inequality. Now, let us consider the characteristic $p = 11$. Take the modular curve $X_k = X_0(23l_k)$, where l_k is the k-th prime number. By [16, Proposition 4.1.20], we easily compute that the genus of X_k is $g_k = 2l_k + 1$. It is also known that the curve X_k has good reduction modulo p outside 23 and l_k. Moreover, by using [16, Proof of Theorem 4.1.52], we obtain that the number of $\mathbb{F}_{p^2}$-rational points over of the reduction X_k/p modulo p satisfies

$$N_1(X_k(\mathbb{F}_{p^2})) \geqslant \frac{\mu_N(p-1)/12}{\deg \lambda_N} \geqslant 2(p-1)(l_k+1)$$

in the notation of loc. cit. Let us take an integer $n > 1$. There exist two consecutive prime numbers l_k and l_{k+1} such that

$$2(p-1)(l_{k+1}+1) > 2n + 2(2l_{k+1}+1) - 2$$

and

$$2(p-1)(l_k+1) \leqslant 2n + 2(2l_k+1) - 2,$$

i. e.

$$(p-1)(l_{k+1}+1) > n+2l_{k+1} \tag{7}$$

and

$$(p-1)(l_k+1) \leqslant n+2l_k. \tag{8}$$

Let us consider the algebraic function field $F_{k+1}/\mathbb{F}_{p^2}$ associated to the curve X_{k+1} of genus $g_{k+1}=2l_{k+1}+1$ defined over $\mathbb{F}_{p^2}$. We have

$$N_1(F_{k+1}/\mathbb{F}_{p^2}) \geqslant 2(p-1)(l_{k+1}+1) > 2n+4l_{k+1}.$$

As before $l_{k+1} \leqslant (1+\varepsilon(l_k))l_k$, with $\varepsilon(l_k)=l_k^{\alpha-1}$. It is also easy to check that the inequality $2g+1 \leqslant q^{\frac{n-1}{2}}(q^{\frac{1}{2}}-1)$ of Theorem 1 holds when q is a power of 11, which follows from the fact that

$$11^{4l_k+\frac{9}{2}}(11^{\frac{1}{2}}-1) \geqslant 8l_k+3.$$

Thus, for any integer $n \geqslant (p-3)x_\alpha+p-1$, the algebraic function field $F_{k+1}/\mathbb{F}_{p^2}$ satisfies Theorem 1, so

$$\mu_{p^2}^{\mathrm{sym}}(n) \leqslant 2n+2l_{k+1} \leqslant 2n+2(1+\varepsilon(l_k))l_k$$

with $l_k \leqslant \dfrac{n}{p-3}-\dfrac{p-1}{p-3}$ by (8).

We remark that as $l_k \leqslant \dfrac{n}{p-3}$, $\varepsilon(l_k) \leqslant \varepsilon_p(n)=\left(\dfrac{n}{p-3}\right)^{\alpha-1}$, which gives the second inequality of the proposition.

Finally, when $n \to +\infty$, the prime numbers $l_k \to +\infty$, thus both for $p \neq 11$ and $p=11$ the corresponding $\varepsilon_p(n) \to 0$. So in the two cases we obtain

$$M_{p^2}^{sym} \leqslant 2\left(1+\frac{1}{p-3}\right). \qquad \square$$

Remark 8. It is easy to see that the bounds obtained in Proposition 7 are generally better than the best known bounds (v) and (vi) recalled in Theorem 3. Indeed, it is sufficient to consider the asymptotic bounds which are deduced from them and to see that for any prime $p \geqslant 5$ we have $\dfrac{1}{p-3} < \dfrac{p}{p-3+(p-1)\frac{p}{p+1}}$ and $\dfrac{1}{p-3} < \dfrac{2}{p-\frac{33}{16}}$ respectively.

Remark 9. Note that the bounds obtained in [13, Corollary 28] also concern the symmetric tensor rank of multiplication in the finite fields even if it is not mentioned. Indeed, the distinction between $\mu_q^{\mathrm{sym}}(n)$ and $\mu_q(n)$ was exploited only from [14]. So, we can compare our Proposition 7 with Corollary 8 there. Firstly, note that the bounds in [13, Corollary 28] are only valid for $p \geqslant 7$. Moreover, the only bound which is better than our bounds is the asymptotic bound [13, Corollary 28, Bound (vi)] given for an unknown sufficiently large n, contrary to our uniform bound with $\alpha = \frac{2}{3}$ for $n \geqslant exp(\exp(33.217))$.

2.2. The case of prime fields

Proposition 10. *Let $p \geqslant 5$ be a prime number, let x_α be defined as in Theorem 6, and $\varepsilon_p(n)$ as in Proposition 7.*

(i) *If $p \neq 11$, then for any integer $n \geqslant \frac{p-3}{2} x_\alpha + \frac{p+1}{2}$ we have*

$$\mu_p^{\mathrm{sym}}(n) \leqslant 3\left(1 + \frac{\frac{4}{3}(1+\varepsilon_p(n))}{p-3}\right)n - \frac{2(1+\varepsilon_p(n))(p+1)}{p-3}.$$

(ii) *For $p = 11$ and $n \geqslant (p-3)x_\alpha + p - 1 = 8x_\alpha + 10$ we have*

$$\mu_p^{\mathrm{sym}}(n) \leqslant 3\left(1 + \frac{\frac{4}{3}(1+\varepsilon_p(n))}{p-3}\right)n - \frac{4(1+\varepsilon_p(n))(p-1)}{p-3} + 1.$$

(iii) *Asymptotically the following inequality holds for any $p \geqslant 5$:*

$$M_p^{\mathrm{sym}} \leqslant 3\left(1 + \frac{\frac{4}{3}}{p-3}\right).$$

Proof. It suffices to consider the same families of curves as in the proof of Proposition 7.

When $p \neq 11$ we take $X_k = X_0(11 l_k)$, where l_k is the k-th prime number. These curves are defined over $\mathbb{F}_p$, hence, we can consider the associated algebraic function fields $F_k/\mathbb{F}_p$ defined over $\mathbb{F}_p$ and we have $N_1(F_k/\mathbb{F}_{p^2}) = N_1(F_k/\mathbb{F}_p) + 2N_2(F_k/\mathbb{F}_p) \geqslant (p-1)(l_k+1)$, since $F_k/\mathbb{F}_{p^2} = F_k/\mathbb{F}_p \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}$ for any k. Note that the genus of the algebraic function fields $F_k/\mathbb{F}_p$ is also $g_k = l_k$, since the genus is preserved under descent.

Given an integer $n > 1$, there exist two consecutive prime numbers l_k and l_{k+1} such that

$$(p-1)(l_{k+1}+1) > 2n + 2l_{k+1} - 2 \tag{9}$$

and

$$(p-1)(l_k+1) \leqslant 2n + 2l_k - 2. \tag{10}$$

Let us consider the algebraic function field $F_{k+1}/\mathbb{F}_p$ associated to the curve X_{k+1} of genus l_{k+1} defined over $\mathbb{F}_p$. We get

$$N_1(F_{k+1}/\mathbb{F}_p) + 2N_2(F_{k+1}/\mathbb{F}_p) \geqslant (p-1)(l_{k+1}+1) > 2n + 2l_{k+1} - 2.$$

As before $l_{k+1} \leqslant (1+\varepsilon(l_k))l_k$, with $\varepsilon(l_k) = l_k^{\alpha-1}$, and from the proof of the previous proposition we know that the inequality $2g+1 \leqslant q^{\frac{n-1}{2}}(q^{\frac{1}{2}}-1)$

of Theorem 1 holds. Consequently, for any integer $n \geqslant \frac{p-3}{2}x_\alpha + \frac{p+1}{2}$, the algebraic function field $F_{k+1}/\mathbb{F}_p$ satisfies Theorem 1, part ii) since by [5, Theorem 11 (i)] there always exists a non-special divisor of degree $g_{k+1}-1$ for $p \geqslant 5$. So

$$\mu_p^{\mathrm{sym}}(n) \leqslant 3n + 2l_{k+1} \leqslant 3n + 2(1+\varepsilon(l_k))l_k$$

with $l_k \leqslant \frac{2n}{p-3} - \frac{p+1}{p-3}$ by (10). As before, $\varepsilon(l_k) \leqslant \varepsilon_p(n) = (\frac{2n}{p-3})^{\alpha-1}$.

When $p=11$ we use once again the family of curves $X_k = X_0(23l_k)$. They are defined over $\mathbb{F}_p$, hence we can consider the associated algebraic function fields $F_k/\mathbb{F}_p$ over $\mathbb{F}_p$ and we have

$$N_1(F_k/\mathbb{F}_{p^2}) = N_1(F_k/\mathbb{F}_p) + 2N_2(F_k/\mathbb{F}_p) \geqslant 2(p-1)(l_k+1).$$

The genus of the algebraic function fields $F_k/\mathbb{F}_p$ defined over $\mathbb{F}_p$ is also $g_k = 2l_k+1$ since the genus is preserved under descent.

Given an integer $n>1$, there exist two consecutive prime numbers l_k and l_{k+1} such that

$$2(p-1)(l_{k+1}+1) > 2n + 2(2l_{k+1}+1) - 2$$

and

$$2(p-1)(l_k+1) \leqslant 2n + 2(2l_k+1) - 2,$$

i. e.

$$(p-1)(l_{k+1}+1) > n + 2l_{k+1} \tag{11}$$

and

$$(p-1)(l_k+1) \leqslant n + 2l_k. \tag{12}$$

Let us consider the algebraic function field $F_{k+1}/\mathbb{F}_p$ associated to the curve X_{k+1} of genus $g_{k+1} = 2l_{k+1}+1$ defined over $\mathbb{F}_p$. We get

$$N_1(F_{k+1}/\mathbb{F}_p) + 2N_2(F_{k+1}/\mathbb{F}_p) \geqslant \\ \geqslant 2(p-1)(l_{k+1}+1) > 2n + 2(2l_{k+1}+1) - 2.$$

As above $l_{k+1} \leqslant (1+\varepsilon(l_k))l_k$, with $\varepsilon(l_k) = l_k^{\alpha-1}$, and the inequality $2g+1 \leqslant q^{\frac{n-1}{2}}(q^{\frac{1}{2}}-1)$ of Theorem 1 holds. Consequently, for any integer $n \geqslant (p-3)x_\alpha + p - 1$, the algebraic function field $F_{k+1}/\mathbb{F}_p$ satisfies Theorem 1, part ii) since, as before, there exists a non-special divisor of degree $g_{k+1}-1$ by [5, Theorem 11 (i)]. So,

$$\mu_p^{\mathrm{sym}}(n) \leqslant 3n + 2g_{k+1} \leqslant 3n + 2(2l_{k+1}+1) \leqslant 3n + 2(1+\varepsilon)l_k$$

with $l_k \leqslant \frac{n}{p-3} - \frac{p-1}{p-3}$ by (12). We can also bound

$$\varepsilon(l_k) \leqslant \varepsilon_p(n) = \left(\frac{n}{p-3}\right)^{\alpha-1}.$$

Finally, when $n \to +\infty$, the prime numbers $l_k \to +\infty$, thus both for $p \neq 11$ and $p = 11$, $\varepsilon_p(n) \to 0$. So we obtain $M_p^{sym} \leqslant 3\left(1 + \frac{4/3}{p-3}\right)$. $\square$

Remark 11. It is easy to see that the bounds obtained in Proposition 10 are generally better than the best known bounds (iii) and (iv) recalled in Theorem 3. Indeed, it is sufficient to consider the asymptotic bounds which are deduced from them and to see that for any prime $p \geqslant 5$ we have

$$\frac{\frac{4}{3}}{p-3} < \frac{\frac{4}{3}p}{p-3+\frac{2(p-1)p}{p+1}}$$

and $\frac{4/3}{p-3} < \frac{8}{3p-5}$ respectively.

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